Exam 1 MTH 230 Spring 2018 Total Pts:100 2/8/2018

Name:________________________ Total Received:

Show all work for full credit. Write all your solutions on the papers provided.
Do not copy answers from graphing calculator.

1. Find the area of the region enclosed by the parabolas $f(x) = 12 - x^2$ and $g(x) = x^2 - 6$. (6 Pts)

2. Find the area of the region enclosed by the curves $f(y) = y^2 - 4y$ and $g(y) = 2y - y^2$. (6 Pts)

3. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1 - x^2$ and $y = 0$ about x-axis. (7 Pts)

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$ about y-axis. (7 Pts)

5. Set up the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$ about the line $y = -1$. (6 Pts)

6. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$ and $y = 0$ $(6 + 6 = 12$ Pts)
   (a) about the y-axis,
   (b) about the line $x = 3$.

7. Evaluate any EIGHT of the following integrals. (56 Pts)
   DO NOT use calculator for any of the integrals.
   (a) $\int \sin^4 x \cos^3 x \, dx$ (b) $\int x^2 \cos x \, dx$
   (c) $\int x^4 \ln x \, dx$ (d) $\int \tan^4 x \, dx$
   (e) $\int \tan^3 x \sec^5 x \, dx$ (f) $\int e^x \sin x \, dx$
   (g) $\int \sin^5 x \cos^9 x \, dx$ (h) $\int \tan^{-1} x \, dx$
   (i) $\int \tan^2 x \sec^4 x \, dx$ (j) $\int \sin^2 x \cos^2 x \, dx$

Extra: Use trigonometric substitution to integrate $\int_0^3 \frac{dx}{\sqrt{x^2+16}}$.

Elisabeth Roberts

1. $f(x) = 12 - x^2$ $\qquad$ $12 - x^2 = x^2 - 6$

$g(x) = x^2 - 6$ $\qquad$ $18 - 2x^2 = 0$

$2(9 - x^2) = 0$

$x = -3, \quad x = 3$

$$A(x) = \int_{-3}^{3} [(12 - x^2) - (x^2 - 6)]dx = \int_{-3}^{3} (18 - 2x^2)dx = \left[18x - \tfrac{2}{3}x^3\right]\Big|_{-3}^{3} =$$

$$= (54 - 18) - (-54 - (-18)) = 36 - (-36) = \boxed{72}$$

2. $f(y) = y^2 - 4y$ $\qquad$ $y^2 - 4y = 2y - y^2$

$g(y) = 2y - y^2$ $\qquad$ $2y^2 - 6y = 0$

$2y(y - 3) = 0$

$y = 0, \ y = 3$

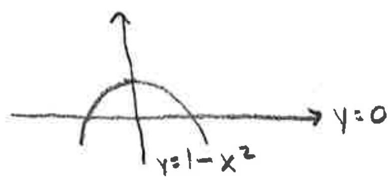
$$A(y) = \int_{0}^{3} [(2y - y^2) - (y^2 - 4y)]dx = \int_{0}^{3} (6y - 2y^2)dx = \left[3y^2 - \tfrac{2}{3}y^3\right]\Big|_{0}^{3} =$$

$$= (27 - 18) - (0) = \boxed{9}$$

3. $y = 1 - x^2$

$y = 0$

about x-axis



$r = 1 - x^2$ $\qquad$ $1 - x^2 = 0$

$1 = x^2$

$\sqrt{1} = x$

$\pm 1 = x$

$$A = \pi r^2 = \pi(1 - x^2)^2 = \pi(1 - 2x^2 + x^4)$$

$$V = \int_{-1}^{1} [\pi(1 - 2x^2 + x^4)]dx = \pi\int_{-1}^{1} (1 - 2x^2 + x^4)dx = \pi\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]\Big|_{-1}^{1} =$$

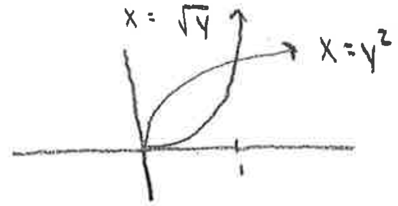
$$= \pi\left[\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - \left(-1 + \tfrac{2}{3} - \tfrac{1}{5}\right)\right] = \pi\left(2 - \tfrac{4}{3} + \tfrac{2}{5}\right) = \pi\left(\frac{30 - 20 + 6}{15}\right) =$$

$$= \pi\left(\tfrac{16}{15}\right) = \boxed{\frac{16\pi}{15}}$$

4. $y = x^2$

$y = \sqrt{x}$

about y-axis



$r_2 = \sqrt{y} = y^{1/2}$

$r_1 = y^2$

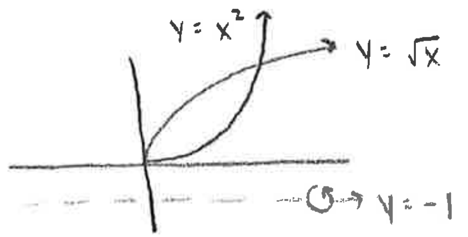
$A(y) = \pi r^2 = \pi\left[(\sqrt{y})^2 - (y^2)^2\right] = \pi(y - y^4)$

$V = \int_0^1 \pi(y - y^4)\,dy = \pi\int_0^1 (y-y^4)\,dy = \pi\left[\frac{1}{2}y^2 - \frac{1}{5}y^5\right]\Big|_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \boxed{\frac{3\pi}{10}}$

5. $y = x^2$

$y = \sqrt{x}$

about $y = -1$



$r_2 = 1 + \sqrt{x}$

$r_1 = 1 + x^2$

$A(x) = \pi r^2 = \pi\left[(1+\sqrt{x})^2 - (1+x^2)^2\right] = \pi\left[(1 + 2x^{1/2} + x) - (1 + 2x^2 + x^4)\right] =$

$= \pi(2x^{1/2} + x - 2x^2 - x^4)$

$V = \int_0^1 \pi(2x^{1/2} + x - 2x^2 - x^4)\,dx = \pi\int_0^1 (2x^{1/2} + x - 2x^2 - x^4)\,dx =$

$= \pi\left[\frac{4}{3}x^{3/2} + \frac{1}{2}x^2 - \frac{2}{3}x^3 - \frac{1}{5}x^5\right]\Big|_0^1 = \pi\left(\frac{4}{3} + \frac{1}{2} - \frac{2}{3} - \frac{1}{5}\right) = \pi\left(\frac{40+15-20-6}{30}\right) =$

$= \pi\left(\frac{29}{30}\right) = \boxed{\frac{29\pi}{30}}$

6a. $y = 2x - x^2$

$y = 0$

about y-axis

$r = x$

$h = 2x - x^2$

$2x - x^2 = 0$

$x(2-x) = 0$

$x = 0,\ x = 2$

$V = \int_0^2 2\pi r h\,dx = 2\pi\int_0^2 x(2x - x^2)\,dx = 2\pi\int_0^2 (2x^2 - x^3)\,dx =$

$= 2\pi\left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]\Big|_0^2 = 2\pi\left(\frac{16}{3} - 4\right) = 2\pi\left(\frac{4}{3}\right) = \boxed{\frac{8\pi}{3}}$

6b. $y = 2x - x^2$

$y = 0$

about $x = 3$

$r = 3 - x$

$h = 2x - x^2$

$x = 0,\ x = 2$

$V = \int_0^2 2\pi(3-x)(2x - x^2)\,dx = 2\pi\int_0^2 \overbrace{(6x - 3x^2 - 2x^2 + x^3)}^{6x - 5x^2 + x^3}\,dx =$

$= 2\pi\left[3x^2 - \frac{5}{3}x^3 + \frac{1}{4}x^4\right]\Big|_0^2 = 2\pi\left(12 - \frac{40}{3} + 4\right) = 2\pi\left(\frac{36-40+12}{3}\right) =$

$= 2\pi\left(\frac{8}{3}\right) = \boxed{\frac{16\pi}{3}}$

Elisabeth Roberts

7a. $\int \sin^4 x\cos^3 x\,dx = \int \sin^4 x\cos^2 x\cos x\,dx =$

$u = \sin x$
$du = \cos x\,dx$

$= \int (\sin^4 x)(1-\sin^2 x)\cos x\,dx = \int u^4(1-u^2)\,du =$

$= \int (u^4 - u^6)\,du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \boxed{\frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C}$

7b. $\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx =$

$= x^2\sin x - \left[-2x\cos x - \int -2\cos x\,dx\right] =$

$= x^2\sin x + 2x\cos x + \int -2\cos x\,dx$

$= \boxed{x^2\sin x + 2x\cos x - 2\sin x + C}$

$u = x^2$
$du = 2x\,dx$
$dv = \cos x\,dx$
$v = \sin x$

$u = 2x$
$du = 2dx$
$dv = \sin x\,dx$
$v = -\cos x$

7c. $\int x^4\ln x\,dx = \frac{x^5}{5}(\ln x) - \int \frac{x^5}{5}\left(\frac{1}{x}\right)dx =$

$= \frac{x^5}{5}(\ln x) - \int \frac{1}{5}x^4\,dx =$

$= \boxed{\frac{x^5\ln x}{5} - \frac{1}{25}x^5 + C}$

$u = \ln x$
$du = \frac{1}{x}dx$
$dv = x^4\,dx$
$v = \frac{1}{5}x^5$

7e. $\int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\sec^4 x\sec x\tan x\,dx =$

$u = \sec x$
$du = \sec x\tan x\,dx$

$= \int (\sec^2 x - 1)\sec^4 x\sec x\tan x\,dx = \int (u^2-1)u^4\,du =$

$\sec^2 x = 1 + \tan^2 x$
$\sec^2 x - 1 = \tan^2 x$

$= \int (u^6 - u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \boxed{\frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C}$

7f. $\int e^x\sin x\,dx = -e^x\cos x - \int -\cos x\,e^x\,dx =$

$= -e^x\cos x + \int e^x\cos x\,dx =$

$= -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

$\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$

$2\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x$

$\int e^x\sin x\,dx = \boxed{\frac{1}{2}\left(-e^x\cos x + e^x\sin x\right) + C}$

$u = e^x$
$du = e^x\,dx$
$dv = \sin x\,dx$
$v = -\cos x$

$u = e^x$
$du = e^x\,dx$
$dv = \cos x\,dx$
$v = \sin x$

7g. $\int \sin^5 x \cos^9 x\,dx = \int \sin^4 x \cos^9 x \sin x\,dx =$

$u = \cos x$
$du = -\sin x\,dx$
$\sin^2 = 1 - \cos^2 x$

$= \int (1-\cos^2 x)^2 \cos^9 x \sin x\,dx =$

$= \int (1 - 2\cos^2 x + \cos^4 x)\cos^9 x \sin x\,dx =$

$= -\int (1 - 2u^2 + u^4)u^9\,du = -\int (u^9 - 2u^{11} + u^{13})\,du =$

$= -\left[\frac{1}{10}u^{10} - \frac{1}{6}u^{12} - \frac{1}{14}u^{14}\right] = \boxed{-\frac{\cos^{10} x}{10} + \frac{\cos^{12} x}{6} - \frac{\cos^{14} x}{14} + C}$

7i. $\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x \sec^2 x \sec^2 x\,dx =$

$u = \tan x$
$du = \sec^2 x\,dx$
$\sec^2 x = 1 + \tan^2 x$

$= \int \tan^2 x(1 + \tan^2 x)\sec^2 x\,dx = \int u^2(1+u^2)\,du =$

$= \int (u^2 + u^4)\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \boxed{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$

7j. $\int \sin^2 x \cos^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\,dx =$

$= \frac{1}{4}\int (1 - \cos^2 2x)\,dx = \frac{1}{4}\int 1 - \frac{1}{2}(1 + \cos 4x)\,dx =$

$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right)dx = \frac{1}{4}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx =$

$\frac{1}{4}\int \frac{1}{2}\,dx - \frac{1}{8}\int \cos 4x\,dx = \frac{1}{4}\left(\frac{x}{2}\right) - \frac{1}{8}\left(\frac{\sin 4x}{4}\right) =$

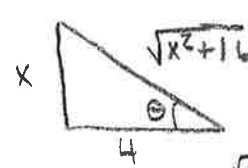
$= \boxed{\frac{x}{8} - \frac{\sin 4x}{32} + C}$

Extra: $\int_0^3 \frac{1}{\sqrt{x^2+16}}\,dx$

$x = 4\tan\theta$
$dx = 4\sec^2\theta\,d\theta$

$\int_0^3 \frac{4\sec^2\theta}{4\sec\theta}\,d\theta = \int_0^3 \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$

$\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right|$



$\ln\left|\frac{\sqrt{25}}{4} + \frac{3}{4}\right| - \ln\left|\frac{\sqrt{16}}{4} + 0\right| =$

$\sec\theta = \frac{\sqrt{x^2+16}}{4}$
$\tan\theta = \frac{x}{4}$

$\ln\left|\frac{8}{4}\right| + \ln(1) = \boxed{\ln\left|\frac{8}{4}\right|} = \boxed{\ln 2}$

Elisabeth Roberts

7d. $\int \tan^4 x\,dx = \int (\sec^2 x - 1)^2\,dx =$

$= \int \sec^4 x - 2\sec^2 x + 1 = \int \sec^4 x - 2\int \sec^2 x + \int dx =$

$= \int \sec^2 x \sec^2 x\,dx - 2\int \sec^2 x + \int dx$

$\int (1+\tan^2 x)\underline{\sec^2 x\,dx}$ $\quad u = \tan x$

$\int (1+u^2)\,du =$ $\quad du = \sec^2 x\,dx$

$u + \frac{1}{3}u^3 = \tan x + \frac{\tan^3 x}{3}$

$$\boxed{\tan x + \frac{\tan^3 x}{3} - 2\tan x + x + C}$$

7h. $\int \tan^{-1} x\,dx =$

$x\tan^{-1}x - \int x\left(\frac{1}{1+x^2}\right)dx =$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx =$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{1}{u}\,du =$

$= \boxed{x\tan^{-1}x - \frac{1}{2}\left(\ln|1+x^2|\right) + C}$

$u = \tan^{-1}x$

$du = \frac{1}{1+x^2}\,dx$

$dv = dx$

$v = x$

$u = 1+x^2$

$\frac{1}{2}du = x\,dx$

Exam 2 MTH 230 Spring 2018 Total Pts:100 3/8/2018

Name:________________________ Total Received:

Show all work for full credit. Write all your solutions on the papers provided.
Do not copy answers from graphing calculator.

1. Evaluate any **SIX** of the following integrals. (36 Pts)
   DO NOT use calculator for any of the integrals.
   (a) $\int x^3\sqrt{4-x^2}\,dx$ (b) $\int \frac{dx}{\sqrt{x^2-9}}$
   (c) $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$ (d) $\int \sqrt{4-x^2}\,dx$
   (e) $\int \frac{20-x}{x^2-5x-6}\,dx$ (f) $\int \frac{6x}{(x-1)(x-2)(x+2)}\,dx$
   (g) $\int \frac{4x^2+3x+9}{(x+3)(x^2+9)}\,dx$ (h) $\int \frac{3x^2+7x+8}{(x-1)(x+2)^2}\,dx$

2. Determine whether the improper integral converges and, if so, evaluate it. (20 Pts)
   (a) $\int_0^\infty xe^{-x^2}\,dx$ (b) $\int_2^\infty \frac{1}{4+x^2}\,dx$
   (c) $\int_3^7 \frac{1}{\sqrt{x-3}}\,dx$ (d) $\int_2^\infty \frac{1}{x\ln x}\,dx$

3. Calculate the arc length of $y=\frac{x^3}{6}+\frac{1}{2x}$ over $[1,2]$. (7 Pts)

4. Calculate the arc length of $y=\ln(\sec x),\ 0\le x\le \pi/4$. (7 Pts)

5. Find the area of the surface generated by rotating the curve
   $y=\sqrt{4-x^2},\ -2\le x\le 2$, about x-axis. (6 Pts)

6. Express $x=e^{2t},\ y=2e^{-6t}$ in the form of $y=f(x)$. (4 Pts)

7. Parametrize the circle $(x+2)^2+(y-3)^2=16$. (4 Pts)

8. Find parametric equations for the line through (1,2) and (-3,6). (4 Pts)

9. Find the equation of the tangent line to the cycloid $c(t)=(2(t-\sin t),2(1-\cos t))$ at $t=\frac{\pi}{2}$. (6 Pts)

10. Graph the parametric curve $x=2+3\cos 2t,\ y=-3+3\sin 2t,\ 0\le t\le\frac{\pi}{2}$.
    Use the Length Formula to find the length of the curve. (6 Pts)

Extra 5 Pts: For the parametric curve $x=e^t,\ y=te^{-t}$,

find $y'(=\frac{dy}{dx})=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $y''(=\frac{d^2y}{dx^2})=\frac{\frac{d(y')}{dt}}{\frac{dx}{dt}}$.

Find the point at which the tangent line is horizontal.

(Hint: Simplify y' before finding y'')

1a. $\int x^3\sqrt{4-x^2}\,dx$ $\quad$ $\boxed{x = 2\sin\theta\ ,\ dx = 2\cos\theta\, d\theta}$

$$\int 8\sin^3\theta\, 2\cos\theta\, 2\cos\theta\, d\theta = 32\int \sin^3\theta\cos^2\theta\, d\theta =$$

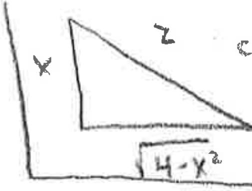
$$= 32\int \sin^2\theta\cos^2\theta\sin\theta\, d\theta =$$

$$= 32\int (1-\cos^2\theta)\cos^2\theta\sin\theta\, d\theta =$$

$\boxed{\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array}}$

$$= -32\int (1-u^2)u^2\, du = -8\int (u^2-u^4)\,du = -8\left(\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right) =$$

$$= -32\left(\frac{\cos^3\theta}{3} - \frac{\cos^5\theta}{5}\right) =$$



$\cos\theta = \frac{\sqrt{4-x^2}}{2}$

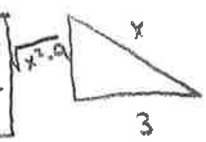
$$= -32\left[\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^3}{3} - \frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^5}{5}\right] =$$

$$= -32\left[\frac{\frac{(\sqrt{4-x^2})^3}{8}}{3} - \frac{\frac{(\sqrt{4-x^2})^5}{32}}{5}\right] = -32\left[\frac{(\sqrt{4-x^2})^3}{24} - \frac{(\sqrt{4-x^2})^5}{160}\right] =$$

$$= \boxed{\frac{-4(\sqrt{4-x^2})^3}{3} + \frac{(\sqrt{4-x^2})^5}{5} + C}$$

1b. $\int \frac{dx}{\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} =$ $\quad$ $\boxed{x = 3\sec\theta,\ dx = 3\sec\theta\tan\theta\, d\theta}$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| = \boxed{\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C}$$



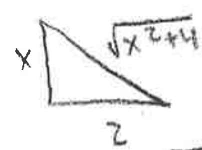
1c. $\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{8\tan^3\theta\, 2\sec^2\theta}{2\sec\theta}\,d\theta$ $\quad$ $\boxed{x = 2\tan\theta,\ dx = 2\sec^2\theta\, d\theta}$

$$= 8\int \tan^3\theta\sec\theta\, d\theta = 8\int \tan^2\theta\sec\theta\tan\theta\, d\theta =$$

$\boxed{\begin{array}{l} 1+\tan^2 x = \sec^2 x \\ u = \sec\theta \\ du = \sec\theta\tan\theta\, d\theta \end{array}}$

$$= 8\int (\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta = 8\int (u^2-1)\,du = 8\left(\tfrac{1}{3}u^3 - u\right) =$$

$$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) = 8\left[\frac{\left(\frac{\sqrt{x^2+4}}{2}\right)^3}{3} - \frac{\sqrt{x^2+4}}{2}\right] =$$



$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

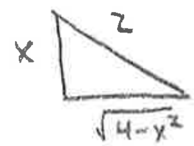
$$= 8\left[\frac{(\sqrt{x^2+4})^3}{24} - \frac{\sqrt{x^2+4}}{2}\right] = \boxed{\frac{(\sqrt{x^2+4})^3}{3} - 4\sqrt{x^2+4} + C}$$

Elisabeth Robert

1d. $\int \sqrt{4-x^2}\,dx = \int 2\cos\theta \, 2\cos\theta \, d\theta =$ $\quad \boxed{x = 2\sin\theta,\ dx = 2\cos\theta\, d\theta}$

$= 4\int \cos^2\theta \, d\theta = 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta = 2\int (1+\cos 2\theta)\,d\theta =$

$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) = 2\theta + 2\sin\theta\cos\theta =$



$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2} \cdot \frac{\sqrt{4-x^2}}{2}\right) =$

$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C}$

1e. $\int \frac{20-x}{x^2-5x-6}\,dx = \int \frac{20-x}{(x-6)(x+1)}\,dx = \frac{A}{x-6} + \frac{B}{x+1} \Big] (x-6)(x+1)$

$20 - x = A(x+1) + B(x-6)$

$x = -1$: $21 = -7B$

$B = -3$

$x = 6$: $14 = 7A$

$A = 2$

$\int \frac{2}{x-6}\,dx + \int \frac{-3}{x+1}\,dx =$

$= \boxed{2\ln|x-6| - 3\ln|x+1| + C}$

Elisabeth Roberts

1f. $\int \frac{6x}{(x-1)(x-2)(x+2)}dx = \left.\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2}\right](x-1)(x-2)(x+2)$

$6x = A(x-2)(x+2) + B(x-1)(x+2) + C(x-1)(x-2)$

$x=2: 12 = 4B$

$B = 3$

$x=-2: -12 = 12C$

$C = -1$

$x=1: 6 = -3A$

$A = -2$

$\int \frac{-2}{x-1}dx + \int \frac{3}{x-2}dx + \int \frac{-1}{x+2}dx =$

$= \boxed{-2\ln|x-1| + 3\ln|x-2| - \ln|x+2| + C}$

1g. $\int \frac{4x^2+3x+9}{(x+3)(x^2+9)}dx = \left.\frac{A}{x+3} + \frac{Bx+C}{x^2+9}\right](x+3)(x^2+9)$

$4x^2+3x+9 = A(x^2+9) + (Bx+C)(x+3)$

$x=-3: \quad 36 = 18A$

$A = 2$

$x=0: \quad 9 = 9A + 3C \rightarrow 9 = 9(2)+3C \rightarrow -9 = 3C \rightarrow C = -3$

$x=1: \quad 16 = 10A + 4B + 4C \rightarrow 16 = 20 + 4B - 12 \rightarrow B = 2$

$\int \frac{2}{x+3}dx + 2\int \frac{x}{x^2+9}dx - 3\int \frac{1}{x^2+9}dx = \quad \left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$\left[\frac{1}{2}\ln|a^2+x^2|\right]$

$2\ln|x+3| + 2\left(\frac{1}{2}\ln|x^2+9|\right) - 3\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) =$

$= \boxed{2\ln|x+3| + \ln|x^2+9| - \tan^{-1}\left(\frac{x}{3}\right) + C}$

1h. $\int \frac{3x^2+7x+8}{(x-1)(x+2)^2}\,dx = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\Big](x-1)(x+2)^2$

$3x^2+7x+8 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$x=-2: \quad 6 = -3C$

$C = -2$

$\int (x+2)^{-2} = -(x+2)^{-1} = \frac{-1}{x+2}$

$x=1: \quad 18 = 9A$

$A = 2$

$x=0: \quad 8 = 4A - 2B - C \to 8 = 8 - 2B + 2 \to B = 1$

$\int \frac{2}{x-1}\,dx + \int \frac{1}{x+2}\,dx - 2\int \frac{1}{(x+2)^2} = \boxed{2\ln|x-1| + \ln|x+2| + \frac{2}{x+2} + C}$

2a. $\int_0^\infty x e^{-x^2}\,dx = \frac{-1}{2}\int_0^\infty e^u\,du =$

$u = -x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\lim_{t\to\infty} e^u = -\frac{1}{2}\lim_{t\to\infty} e^{-x^2} =$

$= -\frac{1}{2}\left[e^{-x^2}\right]\Big|_0^t = \frac{-1}{2}\cdot 0 - \left(\frac{-1}{2}\cdot 1\right) = \boxed{\frac{1}{2} \text{ converges}}$

2b. $\int_2^\infty \frac{1}{4+x^2}\,dx = \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_2^t =$

$\lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \overbrace{\frac{1}{2}\tan^{-1}(1)}^{\pi/4}\right] = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{4} - \frac{\pi}{8} = \boxed{\frac{\pi}{8} \text{ converges}}$

2c. $\int_3^7 \frac{1}{\sqrt{x-3}}\,dx = \lim_{t\to 3^+}\int_t^7 (x-3)^{-1/2}\,dx = \lim_{t\to 3^+} \frac{(x-3)^{1/2}}{1/2} =$

$\lim_{t\to 3^+} 2\sqrt{x-3}\,\Big|_t^7 = 4 - 0 = \boxed{4 \text{ converges}}$

2d. $\int_2^\infty \frac{1}{x\ln x}\,dx = \int_2^\infty \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int_2^\infty u^{-1}\,du =$

$u = \ln x$
$du = \frac{1}{x}dx$

$\lim_{t\to\infty} \ln(\ln|x|)\Big|_2^t = \infty - \ln(\ln|2|) = \boxed{\infty \text{ diverges}}$

$\lim_{t\to\infty}\left[\ln(\ln|t|) - \ln(\ln(2))\right]$

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3. $y = \frac{x^3}{6} + \frac{1}{2x}$ over $[1,2]$ $\quad L = \int_a^b \sqrt{1+(y')^2}$

$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$, $\quad y' = \frac{1}{2}x^2 - \frac{1}{2}x^{-2} = \frac{x^2}{2} - \frac{1}{2x^2}$

$(y')^2 = \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$

$(y')^2 + 1 = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$

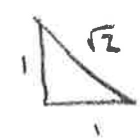
$L = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} = \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)dx = \frac{1}{6}x^3 - \frac{1}{2}x^{-1} =$

$= \left(\frac{x^3}{6} - \frac{1}{2x}\right)\Big|_1^2 = \frac{4}{3} - \frac{1}{4} - \left(\frac{1}{6} - \frac{1}{2}\right) = \frac{16-3-2+6}{12} = \boxed{\frac{17}{12}}$

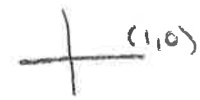
4. $y = \ln(\sec x)$ $\quad 0 \leq x \leq \pi/4$

$y' = \tan x \rightarrow (y')^2 = \tan^2 x \rightarrow (y')^2 + 1 = \tan^2 x + 1 = \sec^2 x$

$L = \int_0^{\pi/4} \sqrt{\sec^2 x} = \int_0^{\pi/4} \sec x\, dx = (\ln|\sec x + \tan x|)\Big|_0^{\pi/4} =$



$= \ln|\sqrt{2}+1| - \ln|1| = \boxed{\ln|\sqrt{2}+1|}$



5. $y = \sqrt{4-x^2}$ $\quad -2 \leq x \leq 2$ $\quad SA = \int 2\pi y\sqrt{1+(y')^2}\,dx$

$y' = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}} \rightarrow (y')^2 = \frac{x^2}{4-x^2}$

$(y')^2 + 1 = \frac{x^2 + 4 - x^2}{4-x^2} = \frac{4}{4-x^2} \rightarrow \sqrt{(y')^2+1} = \frac{2}{\sqrt{4-x^2}}$

$SA = 2\pi \int_{-2}^{2} \sqrt{4-x^2} \cdot \frac{2}{\sqrt{4-x^2}}\, dx = 2\pi \int_{-2}^{2} 2\, dx = 2\pi(2x)\Big|_{-2}^{2} =$

$= (4\pi x)\Big|_{-2}^{2} = 8\pi - (-8\pi) = \boxed{16\pi}$

6. $x = e^{2t}$, $y = 2e^{-6t}$

$(e^{2t})^{-3}$

$y = 2x^{-3}$

7. $(x+2)^2 + (y-3)^2 = 16$

$x+2 = r\cos x \longrightarrow x = -2 + 4\cos t$

$y-3 = r\sin x \longrightarrow y = 3 + 4\sin t$, $0 \le t \le 2\pi$

8. $(1,2)$, $(-3,6)$ $\quad m = \frac{6-2}{-3-1} = \frac{4}{-4} = -1$

$y - 2 = -1(x-1)$

$y - 2 = -x + 1$

$y = -x + 3 \longrightarrow x = t$

$y = -t + 3$

9. $c(t) = (2(t - \sin t), 2(1-\cos t))$ @ $t = \frac{\pi}{2}$

$$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{2\sin t}{2(1-\cos t)}\right|_{t=\frac{\pi}{2}} = \frac{2}{2(1-0)} =$$

$m = \frac{2}{2} = 1$

$x = 2(\frac{\pi}{2} - \sin t)$, $y = 2(1-\cos t)$

$x = 2(\frac{\pi}{2} - 1)$ $\quad y = 2$

$x = \pi - 2$

$(\pi-2, 2)$

$y - 2 = 1(x - (\pi - 2))$

$y - 2 = 1(x - \pi + 2)$

$y = x - \pi + 2 + 2$

$\boxed{y = x - \pi + 4}$

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10. $x = 2 + 3\cos 2t \qquad y = -3 + 3\sin 2t \qquad 0 \le t \le \frac{\pi}{2}$

radius $= 3$ center: $(2, -3)$

$\frac{dx}{dt} = 3(-2\sin 2t) = -6\sin 2t$

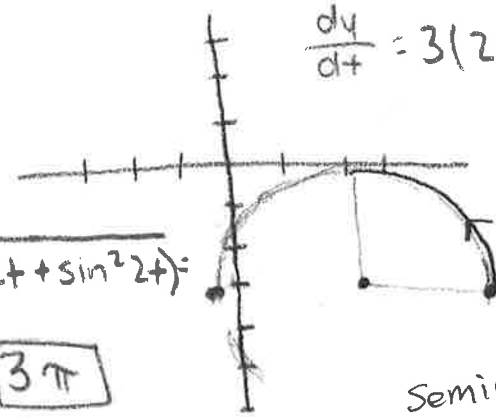
$\frac{dy}{dt} = 3(2\cos 2t) = 6\cos 2t$

$(x-2)^2 + (y+3)^2 = 9$

$L = \int_C^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt =$

$= \int_C^{\pi/2} \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2} = \sqrt{36(\cos^2 2t + \sin^2 2t)} =$

$= \int_0^{\pi/2} \sqrt{36} = \int_0^{\pi/2} 6\,dt = 6t\Big|_0^{\pi/2} = \frac{6\pi}{2} = \boxed{3\pi}$



Semicircle

Extra: $x = e^t \qquad y = te^{-t}$

$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-te^{-t} + e^{-t}}{e^t} = -te^{-2t} + e^{-2t} = \boxed{(-t+1)e^{-2t}}$

$\frac{dy}{dt} = -te^{-t} + e^{-t} = (-t+1)e^{-t}$

$\frac{d(y')}{dt} = -e^{-2t} + (-t+1)(-2)e^{-2t} = (-1+2t-2)e^{-2t} = (2t-3)e^{-2t} \Rightarrow y'' = \frac{(2t-3)e^{-2t}}{e^t}$

$\boxed{y'' = (2t-3)e^{-3t}}$

Tangent is horizontal if $y' = 0 \Rightarrow \frac{dy}{dt} = 0$

$y' = 0 \Rightarrow (-t+1)e^{-2t} = 0$

$t = 1$

The corresponding point on the curve

is $\boxed{\left(e, \frac{1}{e}\right)}$.

Exam 3 MTH 230 Spring 2018 Total Pts:100 4/19/2018

Name:________________________ Total Received:

Show all work for full credit. Write all your solutions on the papers provided.

Class Notes or Phones are not allowed during the exam.

1. Change $P(3, -\sqrt{3})$ to polar coordinate (r, θ) with $r > 0$ and $0 \leq \theta < 2\pi$. Then, find two other representations one with $r > 0$ and the other with $r < 0$. (6 Pts)

2. Change $(\sqrt{2}, \frac{\pi}{4})$ to Cartesian coordinates (x, y). (4 Pts)

3. Convert to rectangular equation: (a) $r = -2$ (b) $r = 2\cos\theta$.
   Sketch the graph. (6 Pts)

4. Sketch the region in the plane consisting of points whose polar coordinates satisfy
   (a) $0 \leq r \leq 3$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ and (b) $r \geq 1$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. (6 Pts)

5. Find the area of the uppper semicircle $r = 2\cos\theta$. (6 Pts)

6. Find the area of one leaf of the "four-petaled rose" $r = \sin 2\theta$. (6 Pts)

7. Sketch the region that lies inside the curve $r = 2\sin\theta$ and outside the curve $r = 1$.
   Find it's area. (8 Pts)

8. Determine whether the sequence converges or diverges by finding the limit. (10 Pts)
   (a) $a_n = \frac{2n^2-4n+6}{n^2-3n+1}$ (b) $b_n = (-1)^n \frac{3n+1}{n+2}$ (c) $c_n = (-1)^{n-1}\frac{n+2}{n^2-2n-3}$ (d) $d_n = \frac{4^n}{(-3)^n}$ (e) $\{\frac{\cos n}{n}\}$

9. Test the series for convergence or divergence. Specify which test or tests you are using by showing the work needed. (36 Pts)

   (a) $\sum_{n=0}^{\infty} \frac{9^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-2n+4}$ (c) $\sum_{n=0}^{\infty} \frac{1}{n^{0.9}+5^n}$

   (d) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{n^2+1}$ (e) $\sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{(n-1)}$ (f) $\sum_{n=0}^{\infty} \frac{3^n+4^n}{5^n}$

   (g) $\sum_{n=0}^{\infty} \frac{2^n}{3^n+6}$ (h) $\sum_{n=0}^{\infty} \left(\frac{2n+1}{3n-2}\right)^n$ (i) $\sum_{n=1}^{\infty} \frac{n^n}{(-3)^{2n}}$

10. Express $2.\overline{32} = 2.3232\cdots$ as a ratio of integers by converting to a geometric series and finding it's sum. (4 Pts)

11. Use the Ratio Test to find the interval of convergence for $f(x) = \sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^n}$. Check at the end points. (8 Pts)

12. **(Extra 5 Pts)** Use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ to find the power series representation for $f(x) = \frac{3}{4-x}$. Find it's interval of convergence.

1. $P(3, -\sqrt{3})$

$x = r\cos\theta \quad y = r\sin\theta \quad \tan\theta = \frac{y}{x}$

$r = \sqrt{(3)^2 + (-\sqrt{3})^2}$

$r = 2\sqrt{3}$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right)$

$\theta = -\frac{\pi}{6} = \frac{11\pi}{6}$

$\boxed{(2\sqrt{3}, \frac{11\pi}{6}) \text{ or } (-2\sqrt{3}, \frac{5\pi}{6}) \text{ or } (2\sqrt{3}, \frac{-\pi}{6})}$

2. $x = \sqrt{2}\cos\frac{\pi}{4} \qquad y = \sqrt{2}\sin\frac{\pi}{4}$

$x = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1 \qquad y = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = 1$

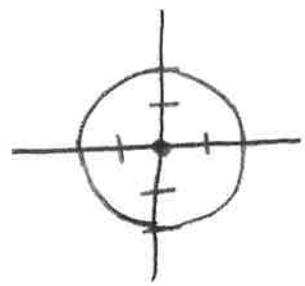
$\boxed{(1, 1)}$

3. A) $r = -2 \qquad r = \sqrt{x^2 + y^2}$

$r^2 = x^2 + y^2$

$(-2)^2 = x^2 + y^2$

$x^2 + y^2 = 4$



B) $r = 2\cos\theta \qquad r = \sqrt{x^2 + y^2}$

$r^2 = x^2 + y^2 \qquad x = r\cos\theta$

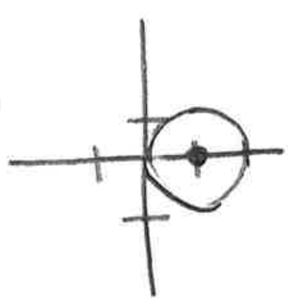
$2\cos\theta = \sqrt{x^2 + y^2}$

$2\left(\frac{x}{r}\right) = \sqrt{x^2 + y^2}$

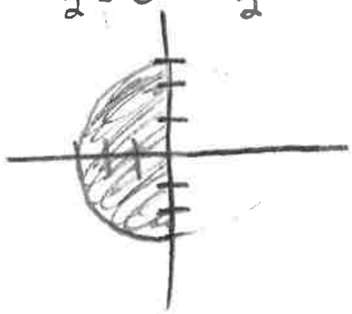
$2x = x^2 + y^2 \quad (r^2 = 2r\cos\theta)$

$x^2 - 2x + 1 + y^2 = 1$

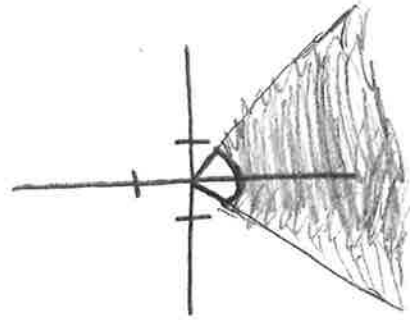
$(x-1)^2 + y^2 = 1$



4. A) $0 \le r \le 3$
$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$



B) $r \ge 1$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$



5. $r = 2\cos\theta$

$A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2}(4\cos^2\theta)\,d\theta = 1\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$

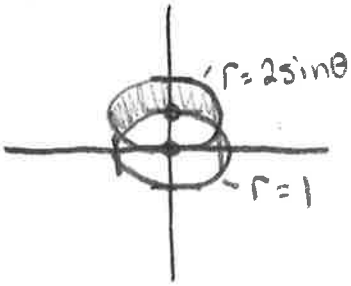
$= \theta + \frac{\sin 2\theta}{2}\Big|_0^{\pi/2} = \boxed{\frac{\pi}{2}\text{ units}^2}$

6. $r = \sin 2\theta$

$A = \int_0^{\pi/2} \frac{1}{2}(\sin^2 2\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\Big|_0^{\pi/2}\right]$

$= \frac{1}{4}\left[\frac{\pi}{2}\right] = \boxed{\frac{\pi}{8}\text{ units}^2}$

7. inside $r = 2\sin\theta$, outside $r = 1$



$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4\sin^2\theta - 1)$

$2\sin\theta = 1$

$= \int_{\pi/6}^{5\pi/6} \frac{1}{2}(2(1-\cos 2\theta) - 1)$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= \int_{\pi/6}^{5\pi/6} 1 - \cos 2\theta - \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$= \int_{\pi/6}^{5\pi/6} \frac{1}{2} - \cos 2\theta$

$= \frac{1}{2}\theta - \frac{\sin 2\theta}{2}\Big|_{\pi/6}^{5\pi/6}$

$= \frac{5\pi}{12} + \frac{\sqrt{3}}{4} - \frac{\pi}{12} + \frac{\sqrt{3}}{4}$

$= \frac{4\pi}{12} + \frac{6\sqrt{3}}{12} = \frac{4\pi + 6\sqrt{3}}{12} = \frac{2\pi + 3\sqrt{3}}{6} = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}\text{ units}^2}$

8. A) $\lim_{n\to\infty} \frac{2n^2-4n+6}{n^2-3n+1} \overset{LH}{\Rightarrow} \lim_{n\to\infty} \frac{4n-4}{2n-3} \overset{LH}{\Rightarrow} \lim_{n\to\infty} \frac{4}{2} \to 2$, so the sequence converges

B) $\lim_{n\to\infty} (-1)^n \frac{3n+1}{n+2} = \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{3n+1}{n+2}$

$\lim_{n\to\infty} \frac{3n+1}{n+2} \overset{LH}{\Rightarrow} \lim_{n\to\infty} \frac{3}{1} \to 3$, so $\lim_{n\to\infty} (-1)^n \frac{3n+1}{n+2} \to \lim_{n\to\infty} (-1)^\infty \cdot 3 = DNE$, so the sequence diverges

C) $\lim_{n\to\infty} (-1)^{n-1} \frac{n+2}{n^2-2n-3} = \lim_{n\to\infty} (-1)^{n-1} \cdot \lim_{n\to\infty} \frac{n+2}{n^2-2n-3}$

$\lim_{n\to\infty} \frac{n+2}{n^2-2n-3} \overset{LH}{\Rightarrow} \lim_{n\to\infty} \frac{1}{2n-2} = \frac{1}{2(\infty)-2} = \frac{1}{\infty} = 0$, so

$\lim_{n\to\infty} (-1)^{n-1} \cdot 0 = 0$, so the sequence converges

D) $\lim_{n\to\infty} = \frac{4^n}{(-3)^n} = \lim_{n\to\infty} (-1)^n \left(\frac{4}{3}\right)^n = (-1)^\infty \cdot \left(\frac{4}{3}\right)^\infty = (-1)^\infty \cdot \infty = DNE$, so the sequence diverges

E) $\lim_{n\to\infty} \frac{\cos n}{n} = \frac{\cos \infty}{\infty} = \frac{-1 \le x \le 1}{\infty} = 0$, so the sequence converges

9. A) $\sum_{n=0}^{\infty} \frac{9^n}{n!}$ $a_{n+1} = \frac{9^n 9}{(n+1)n!}$ Ratio Test

$\left| \frac{9^n 9}{(n+1)n!} \cdot \frac{n!}{9^n} \right| = 9 \cdot \frac{1}{n+1} \to 0 < 1$, so the series converges

B) $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-2n+4}$ $b_n = \frac{2n}{n^3} = \frac{2}{n^2}$ Limit Comparison Test

$\left| \frac{2n+1}{n^3-2n+4} \cdot \frac{n^2}{2} \right| \to 1$ and $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by P-series Test $(p=2)$

so the series converges

C) $\sum_{n=0}^{\infty} \frac{1}{n^{0.9}+5^n}$ Comparison Test

$\frac{1}{n^{0.9}+5^n} \leq \frac{1}{5^n}$ and $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ converges by G. Series Test b/c $|\frac{1}{5}| < 1$

so the series converges

D) $\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{n^2+1}$, $\lim_{n \to \infty} \frac{n+1}{n^2+1} \overset{LH}{=} \lim_{n \to \infty} \frac{1}{2n} = 0$ and

$1 \geq \frac{2}{2} \geq \frac{3}{5} \geq \frac{4}{10} \geq \cdots$

The series converges by the Alt. Series Test.

E) $\sum_{n=2}^{\infty} (-1)^n \frac{n+1}{n-1}$ $\lim_{n\to\infty} \frac{n+1}{n-1} \overset{L'H}{\Rightarrow} \lim_{n\to\infty} \frac{1}{1} \to 1$, so the series

diverges by Alt. Series Test

F) $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$ Geometric Series Test

$\frac{1}{1-\frac{3}{5}} + \frac{1}{1-\frac{4}{5}} = \frac{1}{\frac{2}{5}} + \frac{1}{\frac{4}{5}} = \frac{5}{2} + \frac{5}{4} = \frac{15}{4}$

$\downarrow$ $|r| = \frac{3}{5} < 1$ $\downarrow$ $|r| = \frac{4}{5} < 1$

So the series converges to $\frac{15}{4}$

G) $\sum_{n=0}^{\infty} \frac{2^n}{3^n+6}$ Comparison Test

$\frac{2^n}{3^n+6} \leq \left(\frac{2}{3}\right)^n$ and $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ converges by G. Series Test b/c

$|r| = \frac{2}{3} < 1$, so the series converges

H) $\sum_{n=0}^{\infty} \left(\frac{2n+1}{3n-2}\right)^n$ Root Test

$\left(\left|\frac{2n+1}{3n-2}\right|^n\right)^{\frac{1}{n}} = \frac{2n+1}{3n-2} \to \frac{2}{3} < 1$, so the series converges

absolutely

I) $\sum_{n=1}^{\infty} \frac{n^n}{(9)^n}$ Root Test

$\left(\left|\frac{n}{9}\right|^n\right)^{\frac{1}{n}} = \frac{n}{9} \to \infty > 1$, so the series diverges.

10. $2.3 + 0.023 + 0.00023 + \cdots$

$$= \sum_{n=0}^{\infty} \frac{23}{10}\left(\frac{1}{100}\right)^n = \frac{\frac{23}{10}}{1-\frac{1}{100}} = \frac{\frac{23}{10}}{\frac{99}{100}} = \frac{23}{\cancel{10}} \cdot \frac{\cancel{100}10}{99} = \frac{230}{99}$$

11. $f(x) = \sum_{n=0}^{\infty} \frac{n(x-2)^n}{3^n}$ Ratio Test $a_n = \frac{(n+1)(x-2)^n(x-2)}{3^n 3}$

$$\left|\frac{(n+1)\cancel{(x-2)^n}(x-2)}{\cancel{3^n}3} \cdot \frac{\cancel{3^n}}{n\cancel{(x-2)^n}}\right| = |x-2|\frac{n+1}{3n} \to \frac{|x-2|}{3}$$

At $x=-1$: $\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n n$ $\Rightarrow$ div

At $x=5$: $\sum_{n=0}^{\infty} \frac{n\,3^n}{3^n} = \sum_{n=0}^{\infty} n \Rightarrow$ div

$\frac{|x-2|}{3} < 1$ $\quad \boxed{I = (-1, 5),\ R = 3}$

$|x-2| < 3$

$x-2 < 3$ or $x-2 > -3$

$x < 5$ or $x > -1$

12. $f(x) = \frac{3}{4-x} = \frac{3}{4} \cdot \frac{1}{1-\frac{x}{4}} = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n = \boxed{\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n \quad I = (-4, 4)}$

$\left|\frac{x}{4}\right| < 1$

$\frac{x}{4} < 1$ or $\frac{x}{4} > -1$

$x < 4$ or $x > -4$

10. $2.323232\ldots = \underbrace{2 + 0.32 + 0.0032 + \cdots}_{\text{G.S. with } a = 0.32 = \frac{32}{100},\ r = \frac{1}{100}}$

(OR)

$$= 2 + \frac{\frac{32}{100}}{1-\frac{1}{100}} = 2 + \frac{32}{100}\cdot\frac{100}{99} = 2 + \frac{32}{99}$$

$$= \frac{198+32}{99} = \frac{230}{99}$$

Final Exam MTH 230 Spring 2018 Total Pts: 100 4/27/2018

Name:________________________ Total Received:

Show all work for full credit. Do not use calculator for the integrals. ($10 \times 10 = 100$ *Pts)*

1. Find the Maclaurin series for $f(x) = e^{2x}$.
   Use the Ratio Test to find it's interval of convergence.

   Solve any NINE problems out of the following 14.

2. Find the area of the region enclosed by graphs of $f(x) = x^2 - 3$ and $g(x) = 2x$.
3. Find the volume V obtained by revolving the region between $y = x^2$, $y = \sqrt{x}$ about y-axis.
4. Integrate by the method of Integration by Parts: $\int x^2 e^{x+1}\, dx$.
5. Integrate by the method of Trigonometric Integrals: $\int \tan^5 x \sec^4 x\, dx$
6. Integrate by the method of Trigonometric Substitution: $\int \frac{1}{\sqrt{x^2+9}}\, dx$.
7. Integrate by the method of Partial Fractions: $\int \frac{4x^2+x+5}{(x-1)(x^2+4)}\, dx$.
8. Determine whether the improper integral converges: $\int_2^\infty \frac{1}{x(\ln x)^2}\, dx$.
9. Calculate the arc length of the function $y = \frac{x^3}{6} + \frac{1}{2x}$ over [1,2].
10. Find the equation of the tangent line to the cycloid $c(t) = (t - \sin t, 1 - \cos t)$ at $t = \frac{\pi}{2}$.
11. Find the area of the region that lies inside the cardioid $r = 1 + \sin\theta$ and outside $r = 1$.
12. Determine, with reasons, whether the sequence converges or diverges.
    If it converges, find the limit.
    (a) $a_n = \frac{-2n+1}{n^2+4}$ (b) $b_n = (-1)^n \frac{4+n}{4n^2+1}$ (c) $c_n = \frac{2^n}{\cos^2 n}$
13. (Any TWO) Test the following series by using Div. S. Test/Geo. S. Test/Comparison Test.
    (a) $\sum_{n=0}^{\infty} (-1)^n \frac{n-1}{n+1}$ (b) $\sum_{n=2}^{\infty} \frac{5}{\sin^2 n + 2^n}$ (c) $\sum_{n=0}^{\infty} \frac{3^n - 1}{5^n}$
14. (Any TWO) Test the following series by using Alternating S. Test/Ratio Test/Root Test.
    (a) $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}$ (b) $\sum_{n=1}^{\infty} (\frac{2n+5}{5n+3})^n$ (c) $\sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$
15. Use the Ratio Test to find the interval of convergence for $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n \cdot 2^n}$.

Skylaar Mease

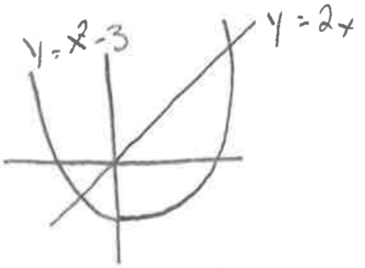
1. $f^{(0)}(x) = e^{2x}$ $\quad f^{(0)}(0) = 2^0$

$f^{(1)}(x) = 2e^{2x}$ $\quad f^{(1)}(0) = 2$

$f^{(2)}(x) = 2^2 e^{2x}$ $\quad f^{(2)}(0) = 2^2$

$$\boxed{\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n, \quad I = (-\infty, \infty)}$$

$$\left| \frac{\cancel{2^n} 2 \cancel{x^n} x}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n} \cancel{x^n}} \right| = \frac{2|x|}{n+1} \to 0 < 1$$

2. 

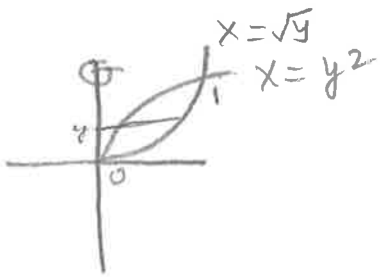
$2x = x^2 - 3$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = -1, 3$

$$A = \int_{-1}^{3} (2x - x^2 + 3)\,dx$$
$$= x^2 - \frac{x^3}{3} + 3x \Big|_{-1}^{3}$$
$$= 9 - \frac{27}{3} + 9 - 1 - \frac{1}{3} + 3 = 20 - \frac{28}{3}$$
$$= \frac{60}{3} - \frac{28}{3} = \frac{32}{3} \text{ units}^2$$

3. 

$r_1 = y^2 \quad r_2 = \sqrt{y}$

$$V = \int_0^1 \pi \left((\sqrt{y})^2 - (y^2)^2 \right) dy = \pi \int_0^1 (y - y^4)\,dy$$
$$= \pi \left[\frac{y^2}{2} - \frac{y^5}{5} \Big|_0^1 \right] = \pi \left[\frac{1}{2} - \frac{1}{5} \right] = \pi \left(\frac{5}{10} - \frac{2}{10} \right)$$
$$= \frac{3\pi}{10} \text{ unit}^3$$

4. $\int x^2 e^{x+1}\,dx$ $\quad u = x^2 \quad du = 2x\,dx$

$dv = e^{x+1}\,dx \quad v = e^{x+1}$

$= x^2 e^{x+1} - \int e^{x+1} 2x\,dx$ $\quad u = 2x \quad du = 2\,dx$

$dv = e^{x+1}\,dx \quad v = e^{x+1}$

$$= x^2 e^{x+1} - 2x e^{x+1} + 2\int e^{x+1}\,dx = \boxed{x^2 e^{x+1} - 2x e^{x+1} + 2e^{x+1} + C}$$

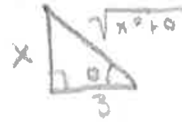
5. $\int \tan^5 x \sec^4 x\,dx$ $\quad u = \tan x$

$du = \sec^2 x\,dx$

$$= \int u^5 (1 + u^2)\,du = \int u^5 + u^7\,du = \boxed{\frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C}$$

$(\sec^2 x = 1 + \tan^2 x)$

6. $\int \frac{1}{\sqrt{x^2+9}}\,dx$ $\quad x = 3\tan\theta$, $\quad dx = 3\sec^2\theta\, d\theta$



$\ln \sec + \tan = \frac{\sec\tan + \sec^2}{\sec + \tan}$

$\frac{\sec(\sec + \tan)}{\sec + \tan}$

$$= \int \frac{3\sec^2\theta\, d\theta}{3\sqrt{\tan^2\theta + 1}} = \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{\cancel{\sec\theta}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C = \boxed{\ln\left| \frac{\sqrt{x^2+9}}{3} + \frac{x}{3} \right| + C}$$

7. $\int \frac{4x^2+x+5}{(x-1)(x^2+4)}\,dx = \int \frac{2}{x-1}\,dx + 2\int \frac{x}{x^2+4}\,dx + 3\int \frac{1}{x^2+4}$

$$= \boxed{2\ln|x-1| + \ln|x^2+4| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

$$\left[\frac{4x^2+x+5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}\right](x-1)(x^2+4)$$

$$4x^2+x+5 = A(x^2+4) + (Bx+C)(x-1)$$

$x=1$: $10 = A(5)$ $\quad A = 2$

$x=0$: $5 = 8 + C(-1)$

$-3 = C(-1)$ $\quad C = 3$

$x=2$: $23 = 2(8) + (2B+3)(1)$

$7 = 2B+3$

$4 = 2B$ $\quad B = 2$

8. $\lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx \quad u = \ln x \quad du = \frac{1}{x}dx$

$= \lim_{t\to\infty}\int_2^t \frac{1}{u^2}\,du = \lim_{t\to\infty}\int_2^t u^{-2}\,du = -\lim_{t\to\infty} u^{-1}\Big|_2^t$

$= -\lim_{t\to\infty} \frac{1}{\ln x}\Big|_2^t = -\lim_{t\to\infty}\left[\frac{1}{\ln t} - \frac{1}{\ln 2}\right] = -\left(\frac{1}{\infty}^{0} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$ Converges

9. $y = \frac{x^3}{6} + \frac{1}{2x}$

$\left(y' = \frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$

$L = \int_1^2 \sqrt{\cancel{\frac{1}{2}} + \frac{x^4}{4} \cancel{-\frac{1}{2}} + \frac{1}{4x^4}}\,dx = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\,dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}x^{-2}\right)dx$

$= \frac{x^3}{6} - \frac{1}{2x}\Big|_1^2 = \frac{8}{6} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{16}{12} - \frac{3}{12} - \frac{2}{12} + \frac{6}{12} = \frac{17}{12}$ units

10. $x = t - \sin t \quad y = 1 - \cos t \qquad \frac{dy}{dx} = \frac{\sin t}{1-\cos t} = \frac{\sin\frac{\pi}{2}}{1-\cos\frac{\pi}{2}} = \frac{1}{1-0} = 1$

$x\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{2}{2} \quad y\left(\frac{\pi}{2}\right) = 1 - 0$

$= \frac{\pi - 2}{2} \qquad = 1$

$y = 1\left(x - \frac{\pi-2}{2}\right) + 1 = x - \frac{\pi-2}{2} + \frac{2}{2} = \boxed{x - \frac{\pi}{2} + 2}$

OR $y = x + \left(2 - \frac{\pi}{2}\right)$

11. inside $r = 1 + \sin\theta$ outside $r = 1$

$1 = 1 + \sin\theta \qquad A = \int_0^\pi \frac{1}{2}\left((1+\sin\theta)^2 - (1)^2\right)d\theta = \frac{1}{2}\int_0^\pi 2\sin\theta + \sin^2\theta\,d\theta$

$\sin^{-1}(0) = \theta \qquad = \int_0^\pi \sin\theta\,d\theta + \frac{1}{2}\int_0^\pi \frac{1}{2}(1 - \cos 2\theta)\,d\theta$

$\theta = 0, \pi \qquad = \left(-\cos\theta + \frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\right)\Big|_0^\pi$

$= -(-1) + \frac{\pi}{4} - (-1) = 2 + \frac{\pi}{4}$ units^2

12. A) $\lim_{n\to\infty} \frac{-2n+1}{n^2+4} \overset{L'H}{\Rightarrow} \lim_{n\to\infty} \frac{-\cancel{2}}{\cancel{2}n} = \frac{1}{\infty} = 0$ converges

B) $\lim_{n\to\infty} \frac{4+n}{4n^2+1} \overset{L'H}{\Rightarrow} \lim_{n\to\infty} \frac{4}{8n} = \frac{4}{\infty} = 0$

$\lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{4+n}{4n^2+1} = (-1)^\infty \cdot 0 = 0$ converges

C) $\lim_{n\to\infty} \frac{2^n}{\cos^2 n} = \lim_{n\to\infty} \frac{2^n}{0 \le x \le 1} = \frac{2^\infty}{0 \le x \le 1} = \infty$ diverges

13. A) $\sum_{n=0}^{\infty} (-1)^n \frac{n-1}{n+1}$ $\quad \lim_{n\to\infty} \frac{n-1}{n+1} \overset{L'H}{\Rightarrow} \lim_{n\to\infty} \frac{1}{1} = 1$

So the series diverges by Alt. Series Test

B) $\sum_{n=2}^{\infty} \frac{5}{\sin^2 n + 2^n} = 5 \sum_{n=2}^{\infty} \frac{1}{\sin^2 n + 2^n}$

$\frac{1}{\sin^2 n + 2^n} \le \left(\frac{1}{2}\right)^n$ and $\sum_{n=2} \left(\frac{1}{2}\right)^n$ converges b/c $\left|\frac{1}{2}\right| < 1$ by

G. Series Test, so the series converges by

Comp. Test

C) $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$

$\left|\frac{3}{5}\right| < 1$ and $\left|\frac{1}{5}\right| < 1$ converge by G. Series Test

converge by G. Series Test

So the series converges by G. Series Test

14. A) $\sum_{n=0}^{\infty} \frac{3^n}{(n+1)!}$ $\quad a_{n+1} = \frac{3^n 3}{(n+2)(n+1)!}$

$\left| \frac{\cancel{3^n}3}{(n+2)\cancel{(n+1)!}} \cdot \frac{\cancel{(n+1)!}}{\cancel{3^n}} \right| = \frac{3}{n+2} \to 0 < 1$

So the series converges by Ratio Test

B) $\sum_{n=1}^{\infty} \left(\frac{2n+5}{5n+3}\right)^n$

$\left(\left|\frac{2n+5}{5n+3}\right|^n\right)^{1/n} = \frac{2n+5}{5n+3} \overset{L'H}{\Rightarrow} \frac{2}{5} \to \frac{2}{5} < 1$

So the series converges by Root Test

C) $\sum_{n=3}^{\infty} (-1)^n \frac{1}{\sqrt{\ln n}}$, $\frac{1}{\sqrt{\ln n}} \to 0$

$f(n) = (\ln n)^{-1/2}$ $\quad f'(n) = -\frac{1}{2}(\ln n)^{-3/2}\frac{1}{n}$

$f'(n) = -\frac{1}{2} \cdot \frac{1}{n(\ln n)^{3/2}} < 0$ for $n \geq 3$

So the series converges by Alt. Series Test

15. $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{n \cdot 2^n}$, $I = (-1, 3]$

$\left| \frac{(-1)^{n+1}\cancel{(x-1)^n}(x-1)}{(n+1)\cdot \cancel{2^n}2} \cdot \frac{n\cancel{2^n}}{(-1)^n \cancel{(x-1)^n}} \right| = \frac{|x-1|}{2}\frac{n}{n+1} \overset{L'H}{\Rightarrow} \frac{|x-1|}{2} \cdot \frac{1}{1} \to \frac{|x-1|}{2} < 1$

$|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

At $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n 2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \to$ div ($p=1$)

At $x=3$: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ Conv by Alt. series test as $\lim_{n\to\infty} \frac{1}{n} = 0$ & $1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \cdots$