

Show all work for full credit.

Name: _____

1. Find an equation of the sphere that passes through the point $(-2,1,3)$ and has center $(1,-2,1)$. (4 Pts)
2. For the vectors $\vec{u} = \langle 1, 2, 3 \rangle$ and $\vec{v} = \langle 2, -1, 2 \rangle$, find $\vec{u} + \vec{v}$, $3\vec{u} - 2\vec{v}$. (4 Pts)
3. For the vectors $\vec{u} = \langle 1, 2, 2 \rangle$, $\vec{v} = \langle 2, -1, 2 \rangle$ and $\vec{w} = \langle 3, 0, 2 \rangle$, find the following. (12 Pts)
 - (a) $2\vec{u} + 3\vec{v}$, $\vec{v} - 3\vec{w}$, $|\vec{u}|$, $|\vec{v}|$
 - (b) Unit vector along the vector \vec{u}
 - (c) The angle θ between \vec{u} and \vec{v}
 - (d) The vector projection of \vec{v} onto \vec{u}
 - (e) The cross product $\vec{u} \times \vec{v}$
 - (f) Area of the parallelogram determined by the vectors \vec{u} and \vec{v}
4. Determine whether the points $A(1,1,-1)$, $B(2,0,-3)$, $C(3,-2,1)$, and $D(3,-4,2)$ are coplanar? (6 Pts)
5. Find parametric equations for the line. (12 Pts)
 - (a) The line through the point $(1,-1,2)$ and perpendicular to the plane $3x - 2y + z = 10$.
 - (b) The line through $(1,0,-2)$ and $(2,1,-3)$.
 - (c) The line through $(1,1,2)$ and parallel to the line $x = 2 - 3t$, $y = -1 + 2t$, $z = 2 + 4t$.
6. Find a scalar equation of the plane. (12 Pts)
 - (a) The plane through $A(1,1,-2)$, $B(2,-2,1)$, and $C(1,2,-3)$.
 - (b) The plane through $(1,-2,2)$ and parallel to the plane $3x - 2y + 2z = 2$.
 - (c) The plane through $(3,2,1)$ and perpendicular to the line $x = -1 - 2t$, $y = -2 + t$, $z = 2 - 3t$.
7. For the planes $P_1 : 3x - y - 2z = 1$ and $P_2 : 2x + 3y + z = 8$: (8 Pts)
 - (a) find the angle θ between them,
 - (b) find the parametric equations for the line of intersection of P_1 and P_2 .
8. Find the distance between the parallel planes $3x - 2y + z = 18$ and $-6x + 4y - 2z = 10$. (4 Pts)
9. Find the limit: $\lim_{t \rightarrow 1} \langle \frac{\ln(2-t)}{t-1}, \frac{t-1}{t^2-1}, \frac{e^t}{\cos t} \rangle$. (5 Pts)
10. For the helix $\vec{r}(t) = \langle t, \sin 2t, -\cos 2t \rangle$, find $\vec{r}'(t)$, $|\vec{r}'(t)|$, $\vec{T}(t)$, $\vec{T}'(t)$, $|\vec{T}'(t)|$, $\vec{N}(t)$, $\vec{B}(t)$, $\vec{T}(\frac{\pi}{2})$, $\vec{N}(\frac{\pi}{2})$, and $\vec{B}(\frac{\pi}{2})$ ($\vec{B} = \vec{T} \times \vec{N}$). Use these to find the curvature κ and osculating plane of the helix $\vec{r}(t)$ at $P(\frac{\pi}{2}, 0, 1)$. (14 Pts)
11. Find parametric equations of the tangent line to $\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 4 \cos 2t \rangle$ at $(\sqrt{3}, 1, 2)$. (5 Pts)
12. Find the velocity and position vectors of a particle that has acceleration $\vec{a} = \langle 6t, t+2, 3 \rangle$ with velocity $\vec{v}(0) = \langle 1, -2, 2 \rangle$ and position $\vec{r}(1) = \langle 1, 1, -1 \rangle$. (6 Pts)
13. Determine whether each statement is true or false in \mathbb{R}^3 ? (8 Pts)
 - (a) Two lines parallel to a plane are parallel.
 - (b) Two planes perpendicular to a line are parallel.
 - (c) Two lines in \mathbb{R}^3 either intersect or are parallel.
 - (d) For any vectors \vec{u} and \vec{v} , we have $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$.
 - (e) The dot product $\vec{u} \cdot \vec{v}$ can be zero for nonzero vectors.
 - (f) The curve with vector equation $\vec{r}(t) = \langle 1 + 2t, 2 - 4t^3, 3t \rangle$ is a line.
 - (g) If $\|\vec{r}(t)\| = 1$ for all t , then $\vec{r}'(t)$ is parallel to $\vec{r}(t)$ for all t .
 - (h) The limit of a vector function is obtained by taking limit of each component function.

1. $P(-2, 1, 3)$ and $C(1, -2, 1)$

William Taylor

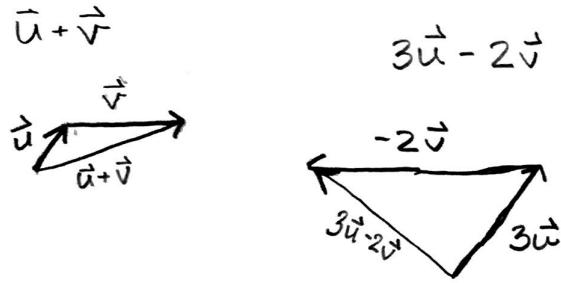
$$\vec{\alpha} = \vec{PC} = \langle 3, -3, -2 \rangle$$

$$r^2 = 22$$

$$r = |\vec{\alpha}| = \sqrt{3^2 + (-3)^2 + (-2)^2}$$
$$= \sqrt{9+9+4} = \sqrt{22}$$

$$(x-1)^2 + (y+2)^2 + (z-1)^2 = 22$$

2. $\vec{u} = \nearrow \quad \vec{v} = \longrightarrow$



3. $\vec{u} = \langle 1, 2, 2 \rangle \quad \vec{v} = \langle 2, -1, 2 \rangle \quad \vec{w} = \langle 3, 0, 2 \rangle$

a) $2\vec{u} + 3\vec{v}$

$$2\langle 1, 2, 2 \rangle + 3\langle 2, -1, 2 \rangle$$

$$\langle 2, 4, 4 \rangle + \langle 6, -3, 6 \rangle$$

$$\boxed{\langle 8, 1, 10 \rangle}$$

$$\vec{v} - 3\vec{w}$$

$$\langle 2, -1, 2 \rangle - 3\langle 3, 0, 2 \rangle$$

$$\langle 2, -1, 2 \rangle - \langle 9, 0, 6 \rangle$$

$$\boxed{\langle -7, -1, -4 \rangle}$$

$$|\vec{u}| = \sqrt{1^2 + 2^2 + 2^2}$$
$$= \sqrt{1+4+4}$$
$$= \sqrt{9}$$

$$\boxed{|\vec{u}| = 3}$$

$$|\vec{v}| = \sqrt{2^2 + (-1)^2 + 2^2}$$
$$= \sqrt{4+1+4}$$
$$= \sqrt{9}$$

$$\boxed{|\vec{v}| = 3}$$

3b.

$$\frac{\vec{u}}{|\vec{u}|}$$

$$\vec{u} = \langle 1, 2, 2 \rangle$$

$$|\vec{u}| = 3$$

3c.

$$\vec{v} = \langle 2, -1, 2 \rangle$$

$$|\vec{v}| = 3$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$$

$$\boxed{\frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle}$$

$$\vec{u} \cdot \vec{v} = (1)(2) + (2)(-1) + (2)(2)$$

$$= 2 - 2 + 4$$

$$= 4$$

$$\cos \theta = \frac{4}{3 \cdot 3}$$

3d. $\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \right) \left(\frac{\vec{u}}{|\vec{u}|} \right)$

$$= \left(\frac{4}{3} \right) \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\boxed{\text{proj}_{\vec{u}} \vec{v} = \left\langle \frac{4}{9}, \frac{8}{9}, \frac{8}{9} \right\rangle}$$

$$\theta = \cos^{-1}(4/9)$$

$$\boxed{\theta \approx 63.6^\circ}$$

3e. $\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \langle 4+2, 2-4, -1-4 \rangle$

\uparrow
make (-)

$$\boxed{\vec{u} \times \vec{v} = \langle 6, -2, -5 \rangle}$$

3f. Area = $|\vec{u} \times \vec{v}|$

$$= \sqrt{6^2 + (-2)^2 + (-5)^2}$$

$$= \sqrt{36 + 4 + 25} = \sqrt{65}$$

$$\boxed{\text{Area} = \sqrt{65} \text{ units}^2}$$

4. A(1, 1, -1)

B(2, 0, -3)

C(3, -2, 1)

D(3, -4, 2)

$\vec{a} = \vec{AB} = \langle 1, -1, -2 \rangle$

$\vec{b} = \vec{AC} = \langle 2, -3, 2 \rangle$

$\vec{c} = \vec{AD} = \langle 2, -5, 3 \rangle$

Volume = $\vec{a} \cdot (\vec{b} \times \vec{c})$

$$V = \begin{vmatrix} 1 & -1 & -2 \\ 2 & -3 & 2 \\ 2 & -5 & 3 \end{vmatrix}$$

$$= (-9+10) + (6-4) - 2(-10+6)$$

$$= 1 + 2 + 8$$

$$V = 11$$

not coplanar

$$1+2+8$$

11 ✓

5a. $P_0(1, -1, 2)$ $\perp 3x - 2y + z = 10$

$$\vec{n} = \langle 3, -2, 1 \rangle$$

$$\vec{v} = \langle a, b, c \rangle$$

$x = 1 + 3t$	$y = -1 - 2t$	$z = 2 + t$
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5b. through $A(1, 0, -2)$ and $B(2, 1, -3)$

$$\vec{v} = \vec{AB} = \langle 1, 1, -1 \rangle \quad P_0(1, 0, -2)$$

$x = 1 + t$	$y = t$	$z = -2 - t$
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5c. through $P_0(1, 1, 2)$ and $\parallel x = 2 - 3t \quad y = -1 + 2t \quad z = 2 + 4t$

$$\vec{v} = \langle -3, 2, 4 \rangle$$

$x = 1 - 3t$	$y = 1 + 2t$	$z = 2 + 4t$
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6a. through: $A(1, 1, -2)$ $\vec{r}_0 = \langle 1, 1, -2 \rangle$
 $B(2, -2, 1)$

$C(1, 2, -3)$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$\vec{a} = \vec{AB} = \langle 1, -3, 3 \rangle \quad 0x + y + z = 0 + 1 - 2$$

$$\vec{b} = \vec{AC} = \langle 0, 1, -1 \rangle$$

$$\vec{n} = \vec{a} \times \vec{b} = \langle 0, -1, 1 \rangle$$

$x - y + z = -1$

6b. through $R(1, -2, 2)$ $\parallel 3x - 2y + 2z = 2$
 $\vec{r}_0 = \langle 1, -2, 2 \rangle$ $\vec{n} = \langle 3, -2, 2 \rangle$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$3x - 2y + 2z = 11$$

$$3x - 2y + 2z = 3 + 4 + 4$$

6c. through $P_0(3, 2, 1)$ $\perp x = -1 - 2t \quad y = -2 + t \quad z = 2 - 3t$

$$\vec{r}_0 = \langle 3, 2, 1 \rangle$$

$$\vec{v} = \langle -2, 1, -3 \rangle$$

$$\vec{v} = \vec{n}$$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$-2x + y - 3z = -6 + 2 - 3$$

$$-2x + y - 3z = -1$$

7. $P_1: 3x - y - 2z = 1 \quad \vec{n}_1 = \langle 3, -1, -2 \rangle$ $\vec{n}_1 \cdot \vec{n}_2 = 6 + (-3) + (-2) = 1$
 $P_2: 2x + 3y + z = 8 \quad \vec{n}_2 = \langle 2, 3, 1 \rangle$

a) $\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} = \frac{1}{\sqrt{14} \cdot \sqrt{14}}$

$$\theta = \cos^{-1}\left(\frac{1}{14}\right)$$

$$\theta \approx 85.9^\circ$$

$$|\vec{n}_1| = \sqrt{3^2 + (-1)^2 + (-2)^2} \\ = \sqrt{9 + 1 + 4} \\ = \sqrt{14}$$

$$|\vec{n}_2| = \sqrt{2^2 + 3^2 + 1^2} \\ = \sqrt{4 + 9 + 1} \\ = \sqrt{14}$$

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = \langle 5, -7, 11 \rangle$$

7b.

$$z=0$$

$$\begin{array}{l} 3x - y = 1 \\ 2x + 3y = 8 \end{array} \rightarrow \begin{array}{l} 9x - 3y = 3 \\ 2x + 3y = 8 \end{array}$$

$$P_0(1, 2, 0) \quad \vec{v} = \langle 5, -7, 11 \rangle$$

$$\begin{array}{l} 2(1) + 3y = 8 \\ 2 + 3y = 8 \end{array} \quad \begin{array}{l} 11x = 11 \\ x = 1 \end{array}$$

$$x = 1 + 5t \quad y = 2 - 7t \quad z = 11t$$

$$x = 1 \quad y = 2 \quad z = 0$$

8. $P_1: 3x - 2y + z = 18 \longrightarrow (0, 0, 18) \text{ or } (\underline{6}, \underline{0}, \underline{0})$ William Taylor

$$P_2: -6x + 4y - 2z = 10$$

a b c $\overset{\curvearrowleft}{d}$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|(-6)(6) + 0 + 0 - 10|}{\sqrt{36 + 16 + 4}} = \frac{46}{\sqrt{56}}$$

$D = \frac{46}{\sqrt{56}} \text{ units}$

9. $\lim_{t \rightarrow 1} \left\langle \frac{\ln(2-t)}{t-1}, \frac{t-1}{t^2-1}, \frac{e^t}{\cos t} \right\rangle$

$$\lim_{t \rightarrow 1} \frac{\ln(2-t)}{t-1} = \left[\frac{0}{0} \right] \stackrel{LH}{=} \frac{\frac{-1}{2-t}}{1} = \lim_{t \rightarrow 1} \frac{-1}{2-t} = -1$$

$$\lim_{t \rightarrow 1} \frac{t-1}{t^2-1} = \left[\frac{0}{0} \right] \stackrel{LH}{=} \lim_{t \rightarrow 1} \frac{1}{2t} = \frac{1}{2}$$

$$\lim_{t \rightarrow 1} \frac{e^t}{\cos t} = \frac{e}{\cos(1)}$$

$= \left\langle -1, \frac{1}{2}, \frac{e}{\cos 1} \right\rangle$

$$10. \quad \vec{r}(t) = \langle t, \sin 2t, -\cos 2t \rangle$$

$$\vec{r}'(t) = \langle 1, 2\cos 2t, 2\sin 2t \rangle$$

$$\begin{aligned} |\vec{r}'(t)| &= \sqrt{1^2 + 4\cos^2 2t + 4\sin^2 2t} \\ &= \sqrt{1 + 4(\cos^2 2t + \sin^2 2t)} \\ &= \sqrt{1+4(1)} \end{aligned}$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\cos 2t, \frac{2}{\sqrt{5}}\sin 2t \right\rangle$$

$$|\vec{r}'(t)| = \sqrt{5}$$

$$\vec{T}'(t) = \left\langle 0, -\frac{4}{\sqrt{5}}\sin 2t, \frac{4}{\sqrt{5}}\cos 2t \right\rangle$$

$$\begin{aligned} |\vec{T}'(t)| &= \sqrt{0 + \left(-\frac{4}{\sqrt{5}}\sin 2t\right)^2 + \left(\frac{4}{\sqrt{5}}\cos 2t\right)^2} \\ &= \sqrt{\frac{16}{5}(\sin^2 2t + \cos^2 2t)} \end{aligned}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left\langle 0, -\sin 2t, \cos 2t \right\rangle$$

$$= \frac{\sqrt{16}}{\sqrt{5}}$$

$$|\vec{T}'(t)| = \frac{4}{\sqrt{5}}$$

$$\vec{B}(t) = \vec{N}(t) \times \vec{T}(t) = - \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -\sin 2t & \cos 2t \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\cos 2t & \frac{2}{\sqrt{5}}\sin 2t \end{vmatrix}$$

make (-1)

$$(\vec{B} = \vec{T} \times \vec{N})$$

$$= \left\langle -\frac{2}{\sqrt{5}}\sin^2 2t - \frac{2}{\sqrt{5}}\cos^2 2t, 0 - \frac{1}{\sqrt{5}}(\cos 2t), + \frac{1}{\sqrt{5}}\sin 2t \right\rangle$$

$\frac{-2}{\sqrt{5}}(\sin^2 2t + \cos^2 2t)$

$$\vec{B}(t) = \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\cos 2t, \frac{1}{\sqrt{5}}\sin 2t \right\rangle$$

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\frac{4}{\sqrt{5}}}{\sqrt{5}} = \frac{4}{5}$$

$$k(t) = \frac{4}{5}$$

$$\vec{n} = \vec{B}(t) = \vec{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$$

$$\vec{n}_0 = \left\langle \frac{\pi}{2}, 0, 1 \right\rangle$$

$$-\frac{\pi}{\sqrt{5}} + 0 + 0$$

$$\vec{n} \cdot \vec{n} = \vec{n} \cdot \vec{n}_0$$

$$\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = \frac{\pi}{\sqrt{5}}$$

$$\text{or } 2x + y = \pi$$

11.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t \rangle$$

William Taylor

$$\begin{aligned}\vec{r}'(t) &= \langle -2\sin t, 2\cos t, -8\sin 2t \rangle \\ \vec{v} &= \vec{r}'\left(\frac{\pi}{6}\right) = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle \\ \vec{r}_0 &= \langle \sqrt{3}, 1, 2 \rangle\end{aligned}$$

$$\begin{aligned}2\cos t &= \sqrt{3} \\ \cos t &= \frac{\sqrt{3}}{2} \\ t &= \frac{\pi}{6}\end{aligned}$$

$$\boxed{x = \sqrt{3} + (-2\sin t)t \quad y = 1 + (2\cos t)t \quad z = 2 + (-8\sin 2t)t} \text{ at } t = \frac{\pi}{6}$$

$$x = \sqrt{3} - t \quad y = 1 + \sqrt{3}t \quad z = 2 - 4\sqrt{3}t$$

$$12. \quad \vec{a} = \vec{r}'''(t) = \langle 6t, t^2+2, 3 \rangle$$

$$\vec{v}(t) = \int \vec{a}(t) dt \rightarrow \langle 3t^2, \frac{1}{2}t^2+2t, 3t \rangle + \vec{c}$$

$$\vec{v}(0) = \langle 1, -2, 2 \rangle$$

$$\langle 0, 0, 0 \rangle + \vec{c} = \langle 1, -2, 2 \rangle$$

$$\vec{c} = \langle 1, -2, 2 \rangle$$

$$\boxed{\vec{v}(t) = \langle 3t^2+1, \frac{t^2}{2}+2t-2, 3t+2 \rangle}$$

$$\vec{r}(t) = \int \vec{v}(t) dt$$

$$= \left\langle t^3+t, \frac{t^3}{6}+t^2-2t, \frac{3t^2}{2}+2t \right\rangle + \vec{c}$$

$$\vec{r}(1) = \langle 1, 1, -1 \rangle$$

$$\langle 1+1, \frac{1}{6}+1-2, \frac{7}{2} \rangle + \vec{c} = \langle 1, 1, -1 \rangle$$

$$\langle 1, 1, -1 \rangle - \langle 2, \frac{-5}{6}, \frac{7}{2} \rangle = \vec{c}$$

$$\vec{c} = \langle -1, \frac{11}{6}, -\frac{9}{2} \rangle$$

$$\boxed{\vec{r}(t) = \left\langle t^3+t-1, \frac{t^3+11}{6}+t^2-2t, \frac{3t^2}{2}+2t-\frac{9}{2} \right\rangle}$$

Exam 2 MTH 231 Summer 2023 Total Pts:100 8/3/2023

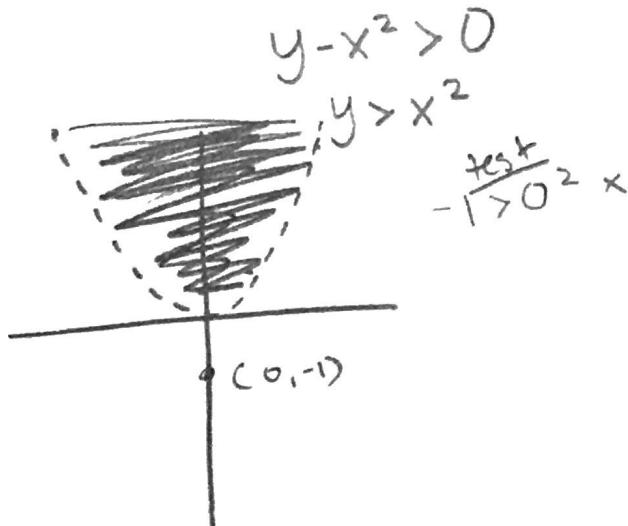
Name: _____

Total Received:

Show all work for full credit.

1. Sketch the domain of the following functions. (6 Pts)
 - a) $f(x, y) = \ln(y - x^2)$
 - b) $g(x, y) = \frac{x^2 + y^2}{\sqrt{x-y+2}}$
2. Show that the following limits do not exist by computing limits along different paths. (6 Pts)
 - (a) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{2x^2 + y^2}$
 - (b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2}$
3. Determine the set of points at which the function $f(x, y) = \frac{xy}{e^{y-x}-1}$ is continuous. (4 Pts)
4. Find f_{xx} , f_{yy} and f_{xy} for the function $f(x, y) = y \sin(x + y)$. (6 Pts)
5. For the function $f(x, y, z) = y \cos(x + yz)$, find f_{yxzx} . (7 Pts)
6. Find an equation of the tangent plane to the surface $f(x, y) = y \cos(2x - y)$ at (1,2,2). (6 Pts)
7. Find the linear approximation of the function $f(x, y) = x \ln(x - y)$ at (4,3) and use it to approximate $f(4.1, 2.9)$. (6 Pts)
8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $e^{xyz} = \frac{y}{x+z}$. (6 Pts)
9. Write out the Chain Rule for $w = f(x, y, z)$ and $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$. (4 Pts)
10. Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $f(x, y) = 2x^2 - 3xy$, $x = e^{s-t}$ and $y = s^2 + 2t^2$ when $s = 1, t = 1$. (6 Pts)
11. Find the equation of the tangent plane to the level surface $x^2 + \frac{y^2}{4} + z^2 = 3$ at the point (1,2,-1). (6 Pts)
12. Find the directional derivative $D_{\vec{v}} f(x, y)$ of the function $f(x, y) = e^x \cos y$ at (0,0) in the direction of vector $\vec{v} = \langle -3, 4 \rangle$ (6 Pts)
13. Find the maximum rate of change of $f(x, y, z) = \frac{y}{x+z}$ at (1,4,1). What is the direction of this maximum rate of change of f ? (8 Pts)
14. Suppose (1,1) is a critical point of a function f with continuous second derivatives. In each case, what can you say about f ? (6 Pts)
 - (a) $f_{xx}(1, 1) = 3, f_{xy}(1, 1) = 4, f_{yy}(1, 1) = 2$
 - (b) $f_{xx}(1, 1) = -3, f_{xy}(1, 1) = -2, f_{yy}(1, 1) = 2$
 - (c) $f_{xx}(1, 1) = 4, f_{xy}(1, 1) = 1, f_{yy}(1, 1) = 2$
 - (d) $f_{xx}(1, 1) = 1, f_{xy}(1, 1) = 3, f_{yy}(1, 1) = 2$
15. Find the critical points of the function $f(x, y) = y^3 - 6y^2 - 2x^3 - 6x^2 + 48x + 20$. Then, use the Second Derivative Test to determine whether they are local minima, local maxima, or saddle points. Find local maximum and local minimum values. (10 Pts)
16. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2y$ subject to the constraint $x^2 + 2y^2 = 6$. (7 Pts)

1a. $f(x,y) = \ln(y-x^2)$



1b. $g(x,y) = \frac{x^2+y^2}{\sqrt{x-y+2}}$

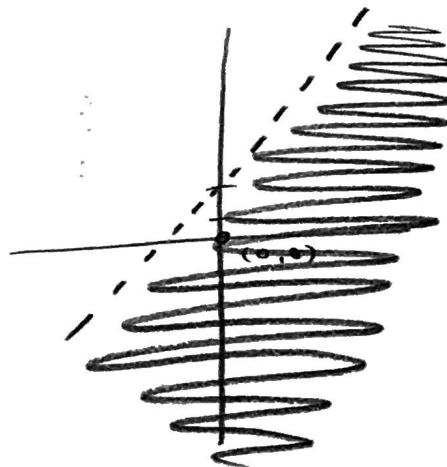
$\sqrt{x-y+2} > 0$

$x-y+2 > 0$

$x+2 > y$

~~$\frac{y+2}{0+2} > 0$~~

$2 > 0$



2a. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2-y^2}{2x^2+y^2}$

along $x=0$

$$\lim_{y \rightarrow 0} \frac{0-y^2}{0+y^2} = -\frac{y^2}{y^2} = -1$$

along $x=y$

$$\lim_{x \rightarrow 0} \frac{x^2-x^2}{2x^2+x^2} = \frac{0}{3x^2} = 0$$

$0 \neq -1$
limit DNE

2b. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4+y^2}$

along $x=0$

$$\lim_{y \rightarrow 0} \frac{0y}{0+y^2} = 0$$

along $y=x^2$

$$\lim_{(x,x^2) \rightarrow (0,0)} \frac{x^2 \cdot x^2}{x^4+(x^2)^2} = \frac{x^4}{2x^4} = \frac{1}{2}$$

$0 \neq \frac{1}{2}$
limit DNE

3. $f(x,y) = \frac{xy}{e^{y-x}-1}$

$$e^{y-x}-1 \neq 0$$

$$e^{y-x} \neq 1$$

$$\downarrow \\ y-x \neq 0$$

$$y \neq x$$

continuous
everywhere
except the
line $y=x$



4. $f(x,y) = y \sin(x+y)$

$$f_x = y \cos(x+y) \cdot 1$$

$$f_y = \sin(x+y) + y \cos(x+y)$$

$$f_{xx} = -y \sin(x+y)$$

$$f_{yx} = f_{xy} = \cos(x+y) - y \sin(x+y)$$

$$f_{yy} = \cos(x+y) + \cos(x+y) + y(-\sin(x+y))$$

$$f_{yy} = 2\cos(x+y) - y \sin(x+y)$$

5. $f(x,y,z) = y \cos(x+yz)$ $\quad f_{yxz} = ?$

$$f_x = -y \sin(x+yz) \cdot 1$$

$$f_{xx} = -y \cos(x+yz)$$

$$f_{xxz} = y^2 \sin(x+yz)$$

$$f_{xxzy} = y^2 z \cos(x+yz) + 2y \sin(x+yz)$$

$$\boxed{f_{yxzx} = f_{xxzy} = y^2 z \cos(x+yz) + 2y \sin(x+yz)}$$

6. $f(x,y) = y \cos(2x-y) \quad @ (1,2,2) \quad z=2$

$$f_x = -2y \sin(2x-y) \quad @ (1,2) \rightarrow = -2(2) \sin(0) = 0$$

$$f_y = -y \sin(2x-y) + \cos(2x-y) \rightarrow = 2 \sin(0) + 1 = 0 + 1 = 1$$

$$z-2 = 0(x-1) + 1(y-2)$$

$$\boxed{z=y}$$

7. $f(x,y) = x \ln(x-y) \quad @ (4,3) \quad z=0$

$$\boxed{x \ln(x-y) \approx (4x-4y-4)}$$

$$f_x = \ln(x-y) + \frac{x}{x-y} \rightarrow \ln(1) + \frac{4}{4-3} = 4$$

$$\begin{aligned} f(4.1, 2.9) &= 4(4.1) - 4(2.9) - 4 \\ &= 16.4 - 11.6 - 4 \\ &= 16.4 - 15.6 \\ &= 0.8 \end{aligned}$$

$$f_y = \frac{-x}{x-y} \rightarrow \frac{-4}{4-3} = -4$$

$$z-0 = 4(x-4) + (-4)(y-3)$$

$$\boxed{f(4.1, 2.9) \approx 0.8}$$

$$z = 4x - 16 - 4y + 12$$

$$z = 4x - 4y - 4$$

$$8. e^{xyz} = \frac{y}{x+z}$$

$$F(x,y,z) = 0 = e^{xyz} - y(x+z)^{-1}$$

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z} = \boxed{- \frac{ye^{xyz} + y(x+z)^{-2}}{xe^{xyz} + y(x+z)^{-2}}}$$

$$\frac{\partial z}{\partial y} = - \frac{F_y}{F_z} = \boxed{- \frac{xze^{xyz} - (x+z)^{-1}}{xye^{xyz} + y(x+z)^{-2}}}$$

$$9. w = f(x, y, z) \quad x = x(s, t) \quad y = y(s, t) \quad z = z(s, t)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$10. f(x, y) = 2x^2 - 3xy = z \quad x = e^{s-t} \quad y = s^2 + 2t^2 \quad \text{when } s=1, t=1$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = (4x - 3y)(e^{s-t}) + (-3x)(2s) \\ &= 4e^{2(s-t)} - 3e^{s-t}(s^2 + 2t^2) - 6se^{s-t} \end{aligned}$$

$$\left. \frac{\partial z}{\partial s} \right|_{s=1, t=1} = 4e^0 - 3e^0(1^2 + 2 \cdot 1^2) - 6(1)e^0$$

$$= 4 - 9 - 6$$

$$\boxed{\left. \frac{\partial z}{\partial s} \right|_{s=1, t=1} = -11}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = (4x - 3y)(-e^{s-t}) + (-3x)(4t)$$

$$= (-e^{s-t})(4e^{s-t} - 3(s^2 + 2t^2)) - (3e^{s-t})(4t)$$

$$\left. \frac{\partial z}{\partial t} \right|_{\substack{s=1 \\ t=1}} = \left. (-e^{s-t})(4e^{s-t} - 3(s^2 + 2t^2)) - (3e^{s-t})(4t) \right|_{\substack{s=1 \\ t=1}}$$

5 - 12

$$\boxed{\left. \frac{\partial z}{\partial t} \right|_{\substack{s=1 \\ t=1}} = -7}$$

11. $x^2 + \frac{y^2}{4} + z^2 = 3$ at $(1, 2, -1)$

$$F(x, y, z) = k$$

$$F_x = 2x + 0 + 0 = 2x$$

$$F_y = \frac{1}{2}y + 0 + 0 = \frac{1}{2}y$$

$$F_z = 2z + 0 + 0 = 2z$$

$$\vec{n} = \nabla f(x, y) = \langle 2x, \frac{1}{2}y, 2z \rangle$$

at $(1, 2, -1)$



$$\vec{n} = \langle 2, 1, -2 \rangle$$

$$\vec{n}_0 = \langle 1, 2, -1 \rangle$$

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

$$2x + y - 2z = 2 + 2 + 2$$

$$\boxed{2x + y - 2z = 6}$$

12. $f(x, y) = e^x \cos y$ at $(0, 0)$ in direction $\vec{v} = \langle -3, 4 \rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$= \langle 1, 0 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle$$

$$= -\frac{3}{5} + 0$$

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$f_x = e^x \cos y \xrightarrow{e^{(0,0)}} e^0 \cos 0 = 1$$

$$f_y = -e^x \sin y \xrightarrow{-e^0 \sin 0} 0$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle -3, 4 \rangle \cdot \frac{1}{\sqrt{9+16}} = \frac{1}{5} \sqrt{25} = s$$

$$\vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$$

$$\nabla f(0, 0) = \langle 1, 0 \rangle$$

$$\boxed{D_{\vec{u}} f = -\frac{3}{5}}$$

$$13. f(x,y,z) = \frac{y}{x+z} = y(x+z)^{-1} \text{ @ } (1,4,1)$$

max rate = $|\nabla f|$

direction = ∇f

$$|\nabla f(1,4,1)| = \sqrt{1 + \frac{1}{4} + 1} = \sqrt{\frac{9}{4}}$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_x = -y(x+z)^{-2} \quad \text{at } (1,4,1) \rightarrow = \frac{-1}{(1+1)^2} = -1$$

$$f_y = (x+z)^{-1} \rightarrow = (1+1)^{-1} = \frac{1}{1+1} = \frac{1}{2}$$

$$f_z = -y(x+z)^{-2} \rightarrow = -1(1+1)^{-2} = -1$$

$$\nabla f(1,4,1) = \left\langle -1, \frac{1}{2}, -1 \right\rangle$$

Max rate of change:

$$|\nabla f| = \frac{3}{2}$$

Direction of:

$$\nabla f = \left\langle -1, \frac{1}{2}, -1 \right\rangle$$

$$14. D = f_{xx}f_{yy} - f_{xy}^2$$

a) $f_{xx} = 3$
 $f_{yy} = 2$
 $f_{xy} = 4$ $D = (3)(2) - (4)^2$
 $= 6 - 16$
 $= -10 < 0$ Saddle point

b) $f_{xx} = -3$
 $f_{yy} = 2$
 $f_{xy} = -2$ $D = (-3)(2) - (-2)^2$
 $= -6 - 4$
 $= -10 < 0$ Saddle point

c) $f_{xx} = 4$
 $f_{yy} = 2$
 $f_{xy} = 1$ $D = (4)(2) - (1)^2$
 $= 8 - 1$
 $= 7 > 0$ local minimum

$$f_{xx} = 4 > 0$$

d) $f_{xx} = 1$
 $f_{yy} = 2$
 $f_{xy} = 3$ $D = (1)(2) - (3)^2$
 $= 2 - 9$
 $= -7 < 0$ Saddle point

$$15. f(x,y) = y^3 - 6xy^2 - 2x^3 - 6x^2 + 48x - 20$$

William Taylor

$$\begin{aligned}f_x &: 0 - 0 - 6x^2 - 12x + 48 \\&= 48 - 12x - 6x^2 = 0 \\4(2 - 2x - x^2) &= 0 \\8 - 2x - x^2 &= 0 \\x^2 + 2x - 8 &= 0 \\(x-2)(x+4) &\\x = 2 \quad x = -4 &\\(2,0) \quad (-4,0) &\end{aligned}$$

$$\begin{aligned}f_y &= 3y^2 - 12y = 0 \\3y^2 - 12y &= 0 \\3y(y-4) &= 0 \\y = 0 \quad y = 4 &\\(2,4) \quad (-4,4) &\end{aligned}$$

$$\begin{aligned}f_{xx} &= -12 - 12x \\f_{yy} &= 6y - 12 \\f_{xy} &= 0\end{aligned}$$

crit pts.

$$\begin{aligned}(2,0) \\(-4,0) \\(2,4) \\(-4,4)\end{aligned}$$

$$D = f_{xx}f_{yy} - f_{xy}^2$$

$$\begin{aligned}D(2,0) &= (-12 - 24)(-12) - 0 \\&= (-36)(-12) \\&= 432 > 0\end{aligned}$$

$$\text{local max} = f(2,0) = 76$$

$$f_{xx} = -36 < 0$$

$$\begin{aligned}D(-4,0) &= (-12 + 48)(-12) - 0 \\&= (36)(-12) \\&= -432 < 0\end{aligned}$$

saddle point

$$\begin{aligned}D(4,4) &= 432 > 0 \\f_{xx} &= 36 > 0\end{aligned}$$

$$\text{local min} = f(-4,4) = -172$$

$$\begin{aligned}D(2,4) &= -432 < 0 \\f_{xx} &= -36 < 0\end{aligned}$$

saddle point

1b. $f(x,y) = x^2y$ constrained at $x^2 + 2y^2 = 6$

$$\nabla f = \lambda \nabla g$$
$$g(x,y) = k$$

$$f_x = \lambda g_x \rightarrow 2xy = \lambda 2x$$
$$f_y = \lambda g_y \rightarrow 2xy - 2\lambda x = 0$$
$$g(x,y) \rightarrow 2x(y - \lambda) = 0$$
$$x=0 \quad y=\lambda$$

$$x^2 = \lambda 4y$$

$$x^2 = y \cdot 4y$$

$$x^2 = 4y^2$$

$$x^2 + 2y^2 = 6$$

$$4y^2 + 2y^2 = 6$$

$$6y^2 = 6$$

$$y^2 = 1$$

$$y = \pm 1$$

$$x = 0$$

$$2y^2 = 6$$

$$y^2 = 3$$

$$y = \pm \sqrt{3}$$

$$(0, \sqrt{3}), (0, -\sqrt{3})$$

$$y = 1$$

$$x^2 + 2 = 6$$

$$x^2 = 4$$

$$x = \pm 2$$

$$y = -1$$

$$x^2 + 2 = 6$$

$$x^2 = 4$$

$$x = \pm 2$$

$$(2, 1)$$

$$(-2, 1)$$

$$(2, -1)$$

$$(-2, -1)$$

Max at
 $(2, 1), (-2, 1)$
Min at
 $(2, -1), (-2, -1)$

Crit pts.

$$f(2, 1) = 4 \cdot 1 = 4$$

$$f(2, -1) = 4 \cdot -1 = -4$$

$$f(-2, 1) = 4 \cdot 1 = 4$$

$$f(-2, -1) = 4 \cdot -1 = -4$$

$$f(0, \sqrt{3}) = 0$$

$$f(0, -\sqrt{3}) = 0$$

Final Exam MTH 231 Ssummer 2023 Total Pts: 100 8/11/2023

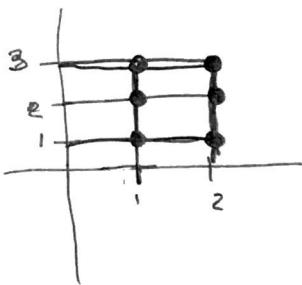
Show all work for full credit.

1. Estimate the volume of the solid that lies below the surface $f(x, y) = x^2 + y^2 + 2$ and above the rectangle $[0, 2] \times [0, 3]$. Use a Riemann sum with $m = 2, n = 3$ and take the sample point to be the upper right corner of each square. (6 Pts)
2. Calculate the double integrals over rectangles. (12 Pts)
 - (a) $\iint_R (x^2 - 2y) dA$, where $R = [0, 2] \times [1, 2]$
 - (b) $\iint_R x \cos(xy) dA$, where $R = \{(x, y) | 1 \leq x \leq 2, 0 \leq y \leq \pi\}$
3. Find the volume of the solid S that is bounded by the elliptic paraboloid $z = x^2 + 2y^2 + 1$, the planes $x = 2$ and $y = 3$ and three coordinate planes. (6 Pts)
4. Calculate the double integrals over general regions. (12 Pts)
 - (a) $\iint_D (x + y) dA$, where D is bounded by $y = 3x, y = 0$, and $x = 3$
 - (b) $\iint_D x \cos y dA$, where D is bounded by $y = 0, y = x^2$ and $x = 1$
5. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -plane bounded by the curves $y = 2x$ and $y = x^2$. (6 Pts)
6. Sketch the region of double integral and change the order of integration. (10 Pts)
 - (a) $\int_0^1 \int_{4x}^4 f(x, y) dy dx$
 - (b) $\int_0^3 \int_{y^2}^9 f(x, y) dx dy$
7. Find the volume of the tetrahedron E bounded by the four planes $x = 0, y = 0, z = 0$, and $x + 2y + z = 4$. (6 Pts)
8. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$. (6 Pts)
9. Evaluate the given integral by changing to polar coordinates: $\iint_R (2y - x) dA$, where R is the region in the third quadrant enclosed by the circle $x^2 + y^2 = 1$ and the lines $x = 0$ and $y = 0$. (6 Pts)
10. Evaluate the double integral $\iint_D e^{x^2+y^2} dA$, where D is the region bounded by the semicircle $x = -\sqrt{4 - y^2}$ and the y -axis. (6 Pts)
11. Evaluate the triple integral $\iiint_E z dV$, where E is bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 2$. (8 Pts)
12. Evaluate $\iiint_E \sqrt{x^2 + z^2} dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$. (8 Pts)
13. Use cylindrical coordinates to evaluate the integral $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 (x^2 + y^2) dz dy dx$. (8 Pts)

$$1. f(x,y) = x^2 + y^2 + 2, \text{ above } R = [0,2] \times [0,3]$$

William Taylor

$$m=2 \quad n=3$$



$$\Delta A = 1$$

$$\Delta A(f(1,1) + f(1,2) + f(2,1) + f(2,2) + f(1,3) + f(2,3))$$

$$(1)(4 + 7 + 7 + 10 + 12 + 15)$$

$$\boxed{\text{Volume} \approx 55 \text{ units}^3}$$

$$2a. \iint_R (x^2 - 2y) dA \quad R = [0, 2] \times [1, 2]$$

$$\begin{aligned} \int_0^2 \int_1^2 x^2 - 2y dy dx &= \int_0^2 [x^2 y - y^2]_1^2 dx = \int_0^2 [2x^2 - 4] - [x^2 - 1] dx \\ &= \int_0^2 x^2 - 3 dx = \left[\frac{1}{3}x^3 - 3x \right]_0^2 = \left[\frac{1}{3} \cdot 2^3 - 6 \right] - 0 = \boxed{\frac{-10}{3}} \end{aligned}$$

$$2b. \iint_R x \cos(xy) dA \quad R = \{(x,y) | 1 \leq x \leq 2, 0 \leq y \leq \pi\}$$

$$\begin{aligned} \int_1^2 \int_0^\pi x \cos(xy) dy dx &= \int_1^2 \left[\cancel{x} \sin(xy) \right]_0^\pi dx = \int_1^2 \sin(\pi x) - \sin 0 dx \\ &= \left[-\frac{1}{\pi} \cos(\pi x) \right]_1^2 = \left[-\frac{1}{\pi} \cos(2\pi) \right] - \left[-\frac{1}{\pi} \cos(\pi) \right] = -\frac{1}{\pi} + \left(-\frac{1}{\pi} \right) = \boxed{-\frac{2}{\pi}} \end{aligned}$$

$$3. V=? \quad D = \{(x,y,z) | 0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq x^2 + 2y^2 + 1\}$$

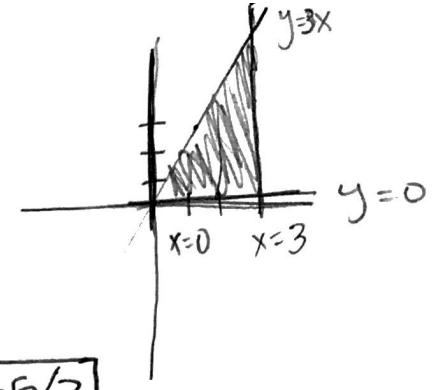
$$V = \iiint_D 1 dV$$

$$\begin{aligned} &= \int_0^2 \int_0^3 \int_0^{x^2+2y^2+1} 1 dz dy dx = \int_0^2 \int_0^3 [z]_0^{x^2+2y^2+1} dy dx = \int_0^2 \int_0^3 (x^2 + 2y^2 + 1) dy dx \\ &= \int_0^2 \left[x^2 y + \frac{2}{3} y^3 + y \right]_0^3 dx = \int_0^2 [3x^2 + 18 + 3] dx = \int_0^2 3x^2 + 21 dx \end{aligned}$$

$$= \left[x^3 + 21x \right]_0^2 = [2^3 + 21 \cdot 2] - 0 = \boxed{50 \text{ units}^3}$$

$$4a. \iint_D (x+y) dA$$

$$D = \begin{cases} y=3x \\ y=0 \\ x=3 \end{cases}$$



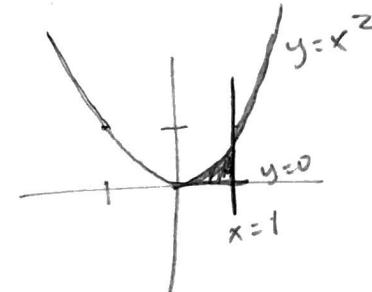
$$D = \{(x,y) \mid 0 \leq x \leq 3, 0 \leq y \leq 3x\}$$

$$= \int_0^3 \int_0^{3x} x+y dy dx = \int_0^3 \left[xy + \frac{1}{2}y^2 \Big|_0^{3x} \right] dx$$

$$= \int_0^3 \left[3x^2 + \frac{9}{2}x^2 \right] - 0 dx = \int_0^3 \frac{15}{2}x^2 dx = \left[\frac{5}{2}x^3 \Big|_0^3 \right] = \boxed{135/2}$$

$$4b. \iint_D x \cos y dA$$

$$D = \begin{cases} y=0 \\ y=x^2 \\ x=1 \end{cases}$$



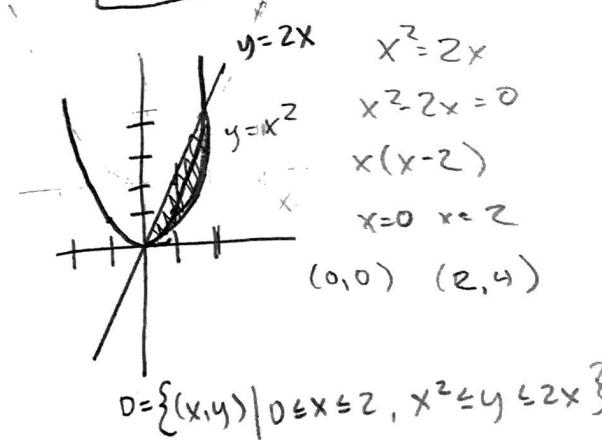
$$\int_0^1 \int_0^{x^2} x \cos y dy dx \quad D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$$

$$= \int_0^1 \left[x \sin y \Big|_0^{x^2} \right] dx = \int_0^1 x \sin x^2 dx = \left[-\frac{\cos x^2}{2} \Big|_0^1 \right] = \left[-\frac{\cos 1}{2} - \left(-\frac{\cos 0}{2} \right) \right]$$

$$= \boxed{\frac{1}{2}(1 - \cos 1)}$$

$$5. V=? = \iiint_D 1 dV \quad 0 \leq z \leq x^2 + y^2$$

$$= \iint_D \int_0^{x^2+y^2} 1 dz dA = \iint_D x^2 + y^2 dA$$



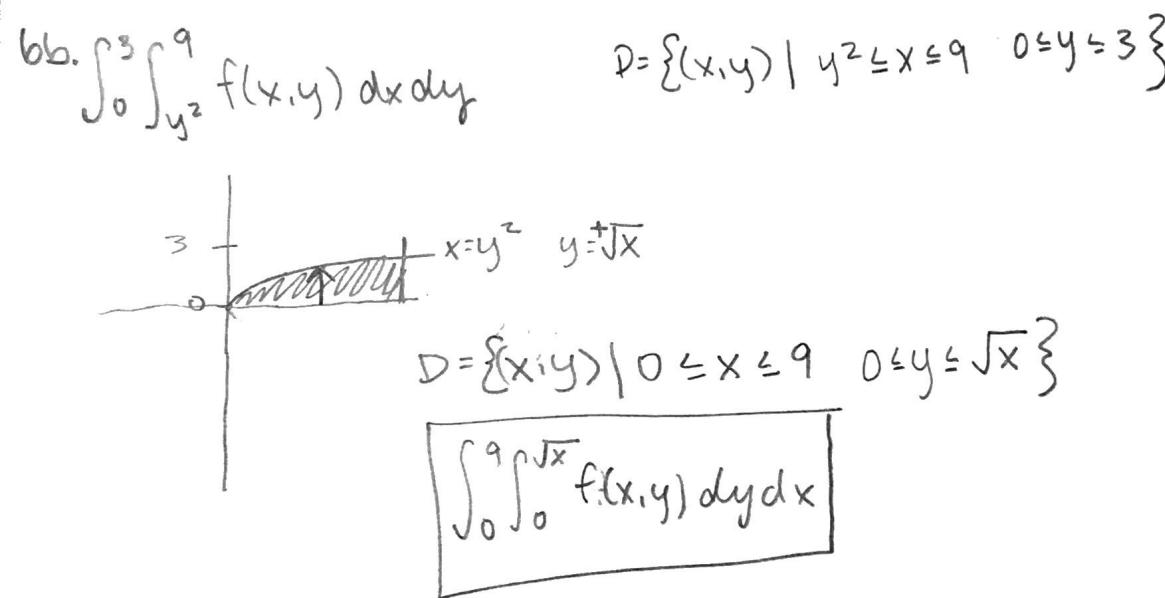
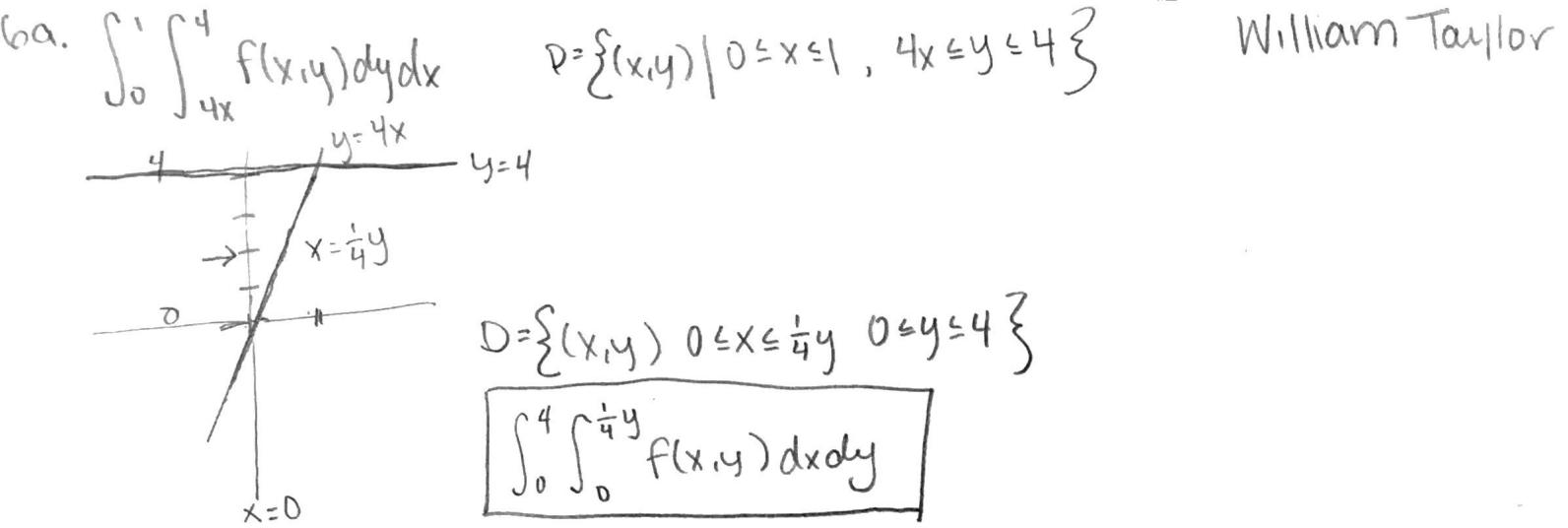
$$= \int_0^2 \int_{x^2}^{2x} x^2 + y^2 dy dx = \int_0^2 \left[x^2 y + \frac{1}{3} y^3 \right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left[x^2 \cdot 2x + \frac{1}{3} (2x)^3 \right] - \left[x^2 \cdot x^2 + \frac{1}{3} (x^2)^3 \right] dx$$

$$= 2x^3 + \frac{8}{3}x^3 - x^4 - \frac{1}{3}x^6 dx$$

$$= \int_0^2 \frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6 dx = \left[\frac{14}{12}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7 \Big|_0^2 \right] = \frac{14}{12}(2)^4 - \frac{1}{5}(2)^5 - \frac{1}{21}(2)^7 - 0$$

$$= \boxed{\frac{216}{35} \text{ units}^3}$$



7. $V = ?$ $0 \leq z \leq 4 - x - 2y$

$$V = \iiint_D 1 dV$$

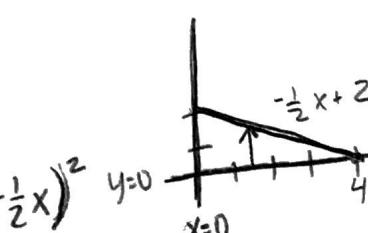
$$V = \iint_D \int_0^{4-x-2y} 1 dz dA = \iint_D 4 - x - 2y dA$$

$$\begin{array}{l} x=0 \\ y=0 \\ 2y+x=4 \\ y=-\frac{1}{2}x+2 \end{array}$$

$$= \int_0^4 \int_0^{2-\frac{1}{2}x} 4 - x - 2y dy dx = \int_0^4 \left[4y - xy - y^2 \right]_0^{2-\frac{1}{2}x} dx$$

$$= \int_0^4 8 - 2x - 2x + \frac{x^2}{2} - (2 - \frac{1}{2}x)^2 - 0 dx = \int_0^4 8 - 4x + \frac{1}{2}x^2 - (2 - \frac{1}{2}x)^2 dx$$

$$= \left[8x - 2x^2 + \frac{1}{6}x^3 + \frac{2(2 - \frac{1}{2}x)^3}{3} \right]_0^4 = \left[32 - 32 + \frac{32}{3} + 0 \right]$$



$$D = \{(x,y) \mid 0 \leq x \leq 4, 0 \leq y \leq -\frac{1}{2}x + 2\}$$

$$- \left[0 + 0 + 0 + \frac{16}{3} \right] = \boxed{\frac{16}{3} \text{ units}^3}$$

$$8. r = \cos 2\theta \quad D = \{(r, \theta) \mid 0 \leq r \leq \cos 2\theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\}$$

$$\begin{aligned} A &= \iint_D 1 \, dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} 1 \cdot r \, dr \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} r^2 \right]_0^{\cos 2\theta} \, d\theta \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cos^2 2\theta \, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} \cdot \frac{1}{2} (1 + \cos 4\theta) \, d\theta \\ &= \frac{1}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + \cos 4\theta \, d\theta = \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \frac{1}{4} \left(\frac{\pi}{4} + 0 \right) - \frac{1}{4} \left(-\frac{\pi}{4} + 0 \right) = \frac{\pi}{16} + \frac{\pi}{16} = \boxed{\frac{\pi}{8} \text{ units}^2} \end{aligned}$$

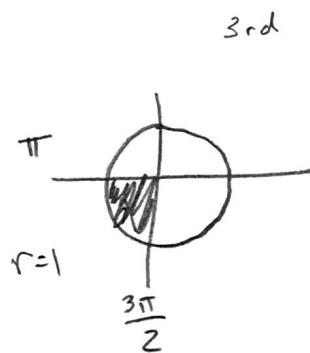


$$9. \iint_R 2y - x \, dA$$

$$R = \begin{cases} x^2 + y^2 = 1 \\ x = 0 \\ y = 0 \end{cases}$$

$$R = \{(r, \theta) \mid 0 \leq r \leq 1, \pi \leq \theta \leq \frac{3\pi}{2}\}$$

$$(2r \sin \theta - r \cos \theta) r \, dr \, d\theta$$

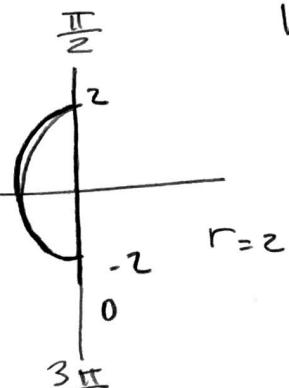


$$\begin{aligned} \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 r^2 (2 \sin \theta - \cos \theta) \, dr \, d\theta &= \int_{\pi}^{\frac{3\pi}{2}} \left[\frac{1}{3} r^3 \right]_0^1 [2 \sin \theta - \cos \theta] \, d\theta \\ &= \frac{1}{3} \int_{\pi}^{\frac{3\pi}{2}} 2 \sin \theta - \cos \theta \, d\theta \\ &= \frac{1}{3} \left[-2 \cos \theta - \sin \theta \Big|_{\pi}^{\frac{3\pi}{2}} \right] \\ &= \frac{1}{3} \left[\cancel{-2 \cos \frac{3\pi}{2}} - \cancel{2 \cos \frac{\pi}{2}} \right] - \frac{1}{3} \left[\cancel{-2 \cos \pi} - \cancel{2 \cos \pi} \right] \\ &= -\frac{1}{3} - \frac{1}{3}(2) = \boxed{-\frac{1}{3}} \end{aligned}$$

3rd

$$10. \iint_D e^{x^2+y^2} dA \quad D = \begin{cases} x = -\sqrt{4-y^2} \\ x = 0 \end{cases}$$

William Taylor



$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^2 e^{r^2} \cdot r dr d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} e^{r^2} \Big|_0^2 d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{1}{2} e^4 - \frac{1}{2} \right] d\theta = \left[\frac{1}{2} e^4 \theta - \frac{1}{2} \theta \right] \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left[\frac{3\pi e^4}{4} - \frac{3\pi}{4} \right] + \left[\frac{-\pi e^4}{4} + \frac{\pi}{4} \right]$$

$$= \frac{\pi(3e^4 - 3 - e^4 + 1)}{4}$$

$$= \boxed{\frac{2\pi e^4 - 2\pi}{4}} = \left(\frac{e^4 - 1}{2} \right) \pi$$

$$11. \iiint_E z dV \quad E = \left\{ \begin{array}{l} x=0 \\ y=0 \\ z=0 \\ x+y+z=2 \end{array} \right\}$$

$$\iiint_D z dz dA \quad \int_D \int_0^{2-x-y} z dz dA \quad 0 \leq z \leq 2-x-y$$

$$= \frac{1}{2} \int_0^2 \int_0^{2-x} (2-x-y)^2 dy dx = \frac{1}{2} \int_0^2 \left[\frac{(2-x-y)^3}{3} \right]_0^{2-x} dx$$

$$= \frac{1}{6} \int_0^2 (2-x-x)^3 - (2-x)^3 dx$$

$$= \frac{1}{6} \int_0^2 (2-x)^3 dx$$

$$= \frac{1}{6} \left[\frac{(2-x)^4}{-4} \right]_0^2$$

$$= \frac{1}{6} \left[\frac{0^4}{-4} \right] - \left[\frac{1}{6} \left[\frac{2^4}{-4} \right] \right] = \boxed{\frac{2}{3}}$$

$$x+y=2$$

$$x=0$$

$$y=0$$



$$D = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2-x\}$$

$$12. \iiint_E \sqrt{x^2 + z^2} \, dV$$

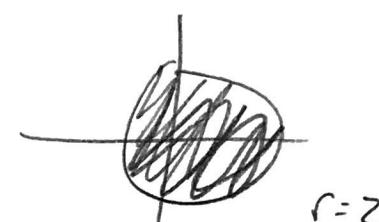
$$E = \begin{cases} y = x^2 + z^2 \\ y = 4 \end{cases}$$

$$\iint_D \int_{x^2+z^2}^4 \sqrt{x^2+z^2} y \, dy \, dA$$

$$x^2 + z^2 \leq y \leq 4$$

$$= \iint_D z \sqrt{x^2 + z^2} \Big|_{x^2+z^2}^4 \, dA$$

xz plane



$$= \iint_D 4\sqrt{x^2 + z^2} - (x^2 + z^2) \sqrt{x^2 + z^2} \, dA = \iint_D 4r - r^3 \, dA$$

$$D = \{(r\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

$$= \int_0^{2\pi} \int_0^2 4r^2 - r^4 \, dr \, d\theta = \int_0^{2\pi} \left[\frac{4}{3}r^3 - \frac{1}{5}r^5 \Big|_0^2 \right] \, d\theta$$

$$r^2 = x^2 + z^2$$

$$= \int_0^{2\pi} \left[\frac{64}{15} - 0 \right] \, d\theta = \left[\frac{64}{15}\theta \right]_0^{2\pi} = \boxed{\frac{128\pi}{15}}$$

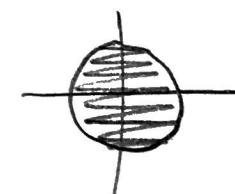
$$13. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 (x^2+y^2) \, dz \, dy \, dx$$

$$\iint_D \int_{\sqrt{x^2+y^2}}^1 (x^2+y^2) \, dz \, dA = \iint_D [z(x^2+y^2)]_{\sqrt{x^2+y^2}}^1 \, dA$$

$$\iint_D (x^2+y^2) - \sqrt{x^2+y^2}(x^2+y^2) \, dA$$

$$y = \sqrt{1-x^2}$$

$$y^2 + x^2 = 1$$



$$r=1$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 1$$

$$\int_0^{2\pi} \int_0^1 [r^2] - [r](r^2) \, dr \, d\theta$$

$$\iint_D r^3 - r^4 \, dr \, d\theta$$

$$\int_0^{2\pi} \left[\frac{1}{4}r^4 - \frac{1}{5}r^5 \Big|_0^1 \right] \, d\theta = \int_0^{2\pi} \left[\frac{1}{4} - \frac{1}{5} \right] \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{20} \, d\theta = \left[\frac{1}{20}\theta \right]_0^{2\pi} = \boxed{\frac{\pi}{10}}$$