Exam 1 MTH 331 Fall 2021 Total Pts:100 9/23/2021

Name:________________________ Total Received:

Show all work for full credit.

1. State True or False. Give short reasons if possible. (10 Pts)
   (a) A system of three linear equations in four unknowns may have a unique solution.
   (b) If two nonzero vectors are LD, then each of them is a scalar multiple of the other.
   (c) The system $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is consistent.
   (d) If $A = [\vec{u}\,\vec{v}\,\vec{w}]$ and $rref(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, then A is invertible.
   (e) The rank of a 3×3 matrix A can be 1.
   (f) For matrices A and B, the formula $A^2B = BA^2$ holds.
   (g) The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ -x + 3y \end{bmatrix}$ is a linear transformation.
   (h) The property $(A + B)C = AC + BC$ holds if the products make sense.
   (i) The column vectors of a matrix $A_{n\times n}$ may not be linearly independent.
   (j) If A and B are matrices of size n, then $rank(A + B) \geq n$ holds.

2. (Paper-pencil) Use G-J elimination method to solve the following linear system. (8 Pts)

$$\begin{aligned} x + y - z &= 7 \\ x - y + 2z &= 3 \\ 2x + y + z &= 9 \end{aligned}$$

3. Consider the linear system (8 Pts)

$$\begin{aligned} y + 2z &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2z &= 2 \end{aligned}$$

   where k is an arbitrary constant. For which value(s) of k does this system have a unique solution or many solutions or no solution? If the system has solution(s), find all such solution(s).

4. Consider the following linear system. (10 Pts)

$$\begin{aligned} x_1 + 2x_3 + 4x_4 &= -8 \\ x_2 - 3x_3 - x_4 &= 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0 \\ -x_2 + 3x_3 + 4x_4 &= -12 \end{aligned}$$

   (i) How many solution(s) do you expect from this system and why?
   (ii) Find "rref" of the augumented matrix.
   (iii) Write the system from part (ii) in terms of x_1, x_2, x_3, x_4, x_5.
   (iv) Write your solutions in vector form using arbitrary constants.

5. The cost of admission to a popular music concert was \$162 for 12 children and 3 adults. The admission was \$122 for 8 children and 3 adults in another music concert. How much was the admission for each child and adult? (8)

6. Write the following linear system into the matrix form $\overrightarrow{y} = A\overrightarrow{x}$. (5 Pts)

$$\begin{aligned} y_1 &= 2x_2 + x_3 - x_4 \\ y_2 &= x_1 - 2x_3 + 3x_4 \\ y_3 &= 3x_1 + 4x_2 + 2x_3 - 3x_4 \\ y_4 &= -x_1 + x_3 - 4x_4. \end{aligned}$$

7. Consider the transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Is this transformation linear? If so, find its matrix. (5 Pts)

8. Given that $proj_L(\overrightarrow{x}) = (\overrightarrow{x}.\overrightarrow{u})\overrightarrow{u}$, and $ref_L(\overrightarrow{x}) = 2proj_L(\overrightarrow{x}) - \overrightarrow{x}$, find the orthogonal projection of the vector $\overrightarrow{x} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ onto the line L which consists of all the scalar multiples of the vector $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Also, find $ref_L(\overrightarrow{x})$. (8 Pts)

9. Compute the matrix product using paper and pencil. (6 Pts)

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

10. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 0 & 3 \\ 4 & 8 & 2 & 2 \end{bmatrix}$. (6 Pts)

11. Show the effects of the matrices (10 Pts)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

on the standard "L" $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$. Describe the transformations in words.

12. Find the inverse of the matrix A using row operations where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$. (8 Pts)

13. Find all linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad \text{(8 Pts)}$$

① a) False It can have infinitely many solutions or no solutionsince there are 3 equations and 4 variables.

b) True Linearly dependent vectors are scalar multiples of other ~~independent vectors.~~

c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ False $0 \neq 1$, so there is no solution.

$\begin{bmatrix} x_1+2x_2+3x_3 \\ 4x_1+5x_2+6x_3 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \text{rref}\left(\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d) True It is only invertible in the form $\text{rref}(A) = I_3$

e) True If a 3x3 matrix in rref only has 1 leading 1, then it can have a rank of 1.

f) False $A^2B = AAB = A(AB) = (AB)A = A(BA) = B(AA) = BA^2$ ~~if~~ $AB = BA$

g) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-3y \\ -x+3y \end{bmatrix}$ True

$T(\vec{x}) = A\vec{x}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-3y \\ -x+3y \end{bmatrix}$

$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} 2x-3y \\ -x+3y \end{bmatrix}$

$ax+by = 2x-3y$ $\quad$ $cx+dy = -x+3y$

$a = 2$ $\quad$ $c = -1$

$b = -3$ $\quad$ $d = 3$ $\quad \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix}$

h) True

i) True The column vectors could be linearly dependent.

j) $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 7 & -5 \\ 1 & -1 \end{bmatrix}$ $\quad A+B = \begin{bmatrix} 9 & 0 \\ -2 & 0 \end{bmatrix}$

$\text{rank}(A) = 2$ $\quad \text{rank}(B) = 2$ $\quad \text{rank}(A+B) = 1$

$\text{rank}(A+B) \leq n$

$1 \leq 2$ False

② $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{array}\right] \xrightarrow[R_3+(-2)R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -2 & 3 & -4 \\ 0 & -1 & 3 & -5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -1 & 3 & -5 \\ 0 & -2 & 3 & -4 \end{array}\right] (-1)R_2$

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & -3 & 5 \\ 0 & -2 & 3 & -4 \end{array}\right] \xrightarrow[R_3+2R_2]{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & -3 & 6 \end{array}\right] \xrightarrow{(-\frac{1}{3})R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} R_1+(-2)R_3 \\ R_2+3R_3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -2 \end{bmatrix}$

③ $\left[\begin{array}{ccc|c} 0 & 1 & 2k & 0 \\ 1 & 2 & 6 & 2 \\ k & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ k & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & -2k & 2-6k & 2-2k \end{array}\right] \begin{array}{l} R_1+(-2)R_2 \\ R_3+2kR_2 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 2-2k & 2-2k \end{array}\right] \xrightarrow{R_3/(2-2k)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1-2R_3 \\ R_2+(-2k)R_3 \end{array}$

$2-2k \neq 0$

$k \neq 1$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2k \\ 0 & 0 & 1 & 1 \end{array}\right]$

unique solution: all values of k except $k=1$

NO solution: Does not occur for this system

inf. many solutions: $k=1$

④ (i) I expect a unique solution or no solution since there are 4 equations and 4 variables. There could be infinitely many solutions.

(ii) $\text{rref}\left(\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right]\right) = \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(iii) $x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$

$x_2 - 3x_3 = 4 \Rightarrow x_2 = 4 + 3x_3$

$x_4 = -2$

(iv) Let $t = x_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ 4+3t \\ t \\ -2 \end{bmatrix} = t\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix}$$

Baile Arkell p.2

(5) c = children

a = adults

$$12c + 3a = 162 \qquad \Rightarrow \text{rref}\left(\left[\begin{array}{cc|c} 12 & 3 & 162 \\ 8 & 3 & 122 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 14 \end{array}\right]$$
$$8c + 3a = 122$$

$$\boxed{\begin{bmatrix} c \\ a \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}}$$

So, admission was \$10 for children and \$14 for adults.

(6)
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 1 & 0 & -2 & 3 \\ 3 & 4 & 2 & -3 \\ -1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(7)
$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$T(\vec{x}) = A\vec{x}$

$$\boxed{A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 5 & 3 \end{bmatrix} \quad \text{Yes, it is linear.}}$$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(8)
$$\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(-\tfrac{8}{3} + 2 + \tfrac{1}{3}\right)\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \left(-\tfrac{1}{3}\right)\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \boxed{\begin{bmatrix} 2/9 \\ -2/9 \\ -1/9 \end{bmatrix} = \text{proj}_L(\vec{x})}$$

$$\text{ref}_L(\vec{x}) = 2\,\text{proj}_L(\vec{x}) - \vec{x} = 2\begin{bmatrix} 2/9 \\ -2/9 \\ -1/9 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -4/9 \\ -2/9 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -32/9 \\ -31/9 \\ -11/9 \end{bmatrix} = \text{ref}_L(\vec{x})}$$

(9) $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$3 \times 3 \quad 3 \times 3$

$$\begin{bmatrix} -1(3)+1(2)+2(1) & -1(1)+1(3)+2(1) & -1(1)+1(1)+2(2) \\ 1(3)+2(2)-2(1) & 1(1)+2(3)-2(1) & 1(1)+2(1)-2(2) \\ 2(3)+3(2)-1(1) & 2(1)+3(3)-1(1) & 2(1)+3(1)-1(2) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 1 & 4 & 4 \\ 5 & 5 & -1 \\ 11 & 10 & 3 \end{bmatrix}}$$

(10) $\text{rref}\left(\begin{bmatrix} 1 & 2 & 4 & -1 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 0 & 3 \\ 4 & 8 & 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

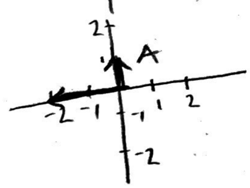
$\boxed{\text{rank} = 3}$

"L"

(11) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



90° counterclockwise rotation

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

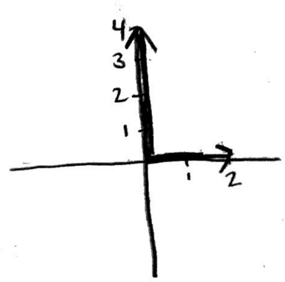
$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

orthogonal projection onto y-axis

$C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$



scaling by a factor of 2

Bailey Arcell P.3

(12) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] R_3+(-5)R_1 \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right] \begin{array}{l} R_1+(-2)R_2 \\ \\ R_3+4R_2 \end{array}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right] \begin{array}{l} R_1+5R_3 \\ R_2+(-4)R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$

$$\boxed{A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}}$$

(13) $T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad T\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$T(\vec{x}) = A\vec{x}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 2a+3b \\ 2c+3d \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$b = 2-a$

$\Uparrow$

$a+b=2$

$c+d=3$

$\Downarrow$

$d=3-c$

$2a+3(2-a)=4$

$2a+6-3a=4$

$-a=-2$

$a=2 \Rightarrow b=2-a$

$=2-2$

$b=0$

$\Uparrow$

$2a+3b=4$

$2c+3d=-1$

$\Downarrow$

$2c+3(3-c)=-1$

$2c+9-3c=-1$

$-c=-10$

$c=10 \Rightarrow d=3-c$

$=3-10$

$d=-7$

$$\boxed{T = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 10 & -7 \end{bmatrix}}$$

Verify:

$\underset{2\times2}{\begin{bmatrix} 2 & 0 \\ 10 & -7 \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ✓

$\begin{bmatrix} 2 & 0 \\ 10 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ ✓

Exam 2 MTH 331 Fall 2021 Total Pts:100 10/25/2021

Name:________________________ Total Received:

Show all work for full credit. Write all your solutions in the papers provided.

1. Find (i) a basis of the kernel and (ii) a basis of the image of the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^4$ given by

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 3 & 1 & 9 & 6 & -8 \\ 1 & -2 & 3 & 1 & -3 \\ 2 & 1 & 6 & 1 & 1 \end{bmatrix} \vec{x}. \qquad \text{(10 Pts)}$$

   What are the dimensions of $ker(T)$ and $im(T)$?

2. Which of the vectors

$$\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \quad \vec{v_4} = \begin{bmatrix} 1 \\ -7 \\ -12 \end{bmatrix}$$

   in $\mathbb{R}^3$ are linearly independent? Find a nontrivial relation among them? (6 Pts)

3. Prove that the set

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0, \text{ and } x, y, z \in \mathbb{R} \right\}$$

   is a subspace of $\mathbb{R}^3$. Find any two bases of the subspace W of $\mathbb{R}^3$.
   State the dimension of W. (10 Pts)

4. Find the redundant column vector(s) of the matrix A where

$$A = \begin{bmatrix} 2 & -1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 2 & 3 \\ -1 & -2 & -3 & 2 & 1 \end{bmatrix}.$$

   Write all possible relationships among the column vectors. (7 Pts)

5. Determine whether the vector $\vec{x}$ is in the span V of the vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ where

$$\vec{x} = \begin{bmatrix} 2 \\ 9 \\ 4 \end{bmatrix}, \quad \vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

   If $\vec{x}$ is in V, write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$. (6 Pts)

6. Is matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ similar to matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$? Show complete work and if they are similar, exhibit the matrix S in the definition ($AS = SB$). (7 Pts)

7. Find a basis for the space of all 2×2 matrices A such that $AB = BA$ where $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and determine its dimension. (6 Pts)

8. Use "column by column" to construct the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, where

$$\vec{v_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}. \text{ (10 Pts)}$$

9. Prove that the subset $W = \{p(x) \in P_2 : p'(1) = p(2)\}$ is a subspace of the space P_2 of all polynomials of degree 2 or less. Find TWO bases of W. What is the dimension of W? (8 Pts)

10. Consider the linear transformation $T(M) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ with standard basis of $\mathbb{R}^{2\times 2}$.
(a) Find the $\mathfrak{B}$-matrix B of the transformation T by using either diagram or column by column.
(b) Find bases of kernel and image of the matrix B.
(c) Find bases of kernel and image of the transformation T (using part (b)).
(d) Determine whether T is isomorphism. (10 Pts)

11. (a) Prove that the transformation $T : P_2 \to P_2$ given by $T(f) = 2f - f'$ is linear.
(b) With the standard basis $\mathcal{B} = (1, x, x^2)$ of P_2, find the $\mathcal{B}$-matrix of the transformation T, by using either the diagram method or the method of the columns of the $\mathcal{B}$-matrix of T.
(c) Determine whether T is isomorphism by analyzing the $\mathcal{B}$-matrix . (10 Pts)

12. State with a brief reason whether the following statements are true or false. (10 Pts)
(a) The kernel of a 4×3 matrix is a subset of $\mathbb{R}^4$.
(b) If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ in $\mathbb{R}^n$ are LD, then there is at least one non-trivial relation among them.
(c) The space P_2 is isomorphic to $\mathbb{R}^3$.
(d) The column vectors of a 4×5 matrix may be linearly independent.
(e) If V and W are subspaces of $\mathbb{R}^n$, then $V \cap W$ is be a subspace of $\mathbb{R}^n$ as well.
(f) The image of a 4×5 matrix A is a subspace of $\mathbb{R}^4$.
(g) If A is a 6×5 matrix of rank 2, then the nullity of A is 3.
(h) The space $\mathbb{R}^{3\times 2}$ is 6-dimensional.
(i) The function $T(f) = 3ff'$ from C^∞ to C^∞ is a linear transformation.
(j) The space of all 2×2 lower-triangular matrices is 3-dimensional.

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MTH 331
10/25/21

① $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 3 & 1 & 9 & 6 & -8 \\ 1 & -2 & 3 & 1 & -3 \\ 2 & 1 & 6 & 1 & 1 \end{bmatrix} \vec{x}$

$$\operatorname{rref}\left(\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 3 & 1 & 9 & 6 & -8 \\ 1 & -2 & 3 & 1 & -3 \\ 2 & 1 & 6 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{v_3} = 3\vec{v_1} \Rightarrow -3\vec{v_1} + \vec{v_3} = \vec{0}$

$\vec{v_5} = \vec{v_1} + \vec{v_2} - 2\vec{v_4}$

$\Rightarrow -\vec{v_1} - \vec{v_2} + 2\vec{v_4} + \vec{v_5} = \vec{0}$

basis of image = ~~span~~ $\left(\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \\ 1 \end{bmatrix}\right)$

$\dim(\operatorname{im}(T)) = 3$

basis of kernel = ~~span~~ $\left(\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$

$\dim(\ker(T)) = 2$

② $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 1 \\ -7 \\ -12 \end{bmatrix}$

$$A = \begin{array}{cccc} \begin{bmatrix} 2 & 1 & 4 & 1 \\ 3 & 2 & -3 & -7 \\ 6 & -1 & 1 & -12 \end{bmatrix} \\ \vec{v_1} \quad \vec{v_2} \quad \vec{v_3} \quad \vec{v_4} \end{array} \qquad \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent

Non-trivial relationship:

$\vec{v_4} = -2\vec{v_1} + \vec{v_2} + \vec{v_3}$

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; x - y + 2z = 0 \text{ and } x, y, z \in \mathbb{R} \right\}$$

(i) zero Element:

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ because $0 - 0 + 2(0) = 0$.

(ii) $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \in W$ because $(x_1 + x_2) - (y_1 + y_2) + 2(z_1 + z_2) = 0$

$\Rightarrow (x_1 + y_1 + 2z_1) + (x_2 - y_2 + 2z_2) = 0$

$\Rightarrow 0 + 0 = 0$

(iii) $k \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$ because $kx - ky + 2kz = 0$

$\Rightarrow k(x - y + 2z) = 0$

$\Rightarrow k(0) = 0$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x - 1 = 0 \quad x + 2 = 0$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \right\}$$

$x - 1 + 2 = 0 \quad x - 2 - 2 = 0$

$x + 1 = 0 \quad x - 4 = 0$

$\dim(W) = 2$

④ $A = \begin{bmatrix} 2 & -1 & 1 & 2 & -1 \\ 1 & 2 & 3 & 2 & 3 \\ -1 & -2 & -3 & 2 & 1 \end{bmatrix}$ $\quad \operatorname{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\ \begin{matrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \vec{v_4} & \vec{v_5} \end{matrix} \end{array}$

$\vec{v_3}$ and $\vec{v_5}$ are redundant vectors.

Relationships among column vectors:

$\vec{v_3} = \vec{v_1} + \vec{v_2}$

$\vec{v_5} = -\vec{v_1} + \vec{v_2} + \vec{v_4}$

⑤ $\vec{x} = \begin{bmatrix} 2 \\ 9 \\ 4 \end{bmatrix}, \vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v_3} = \vec{x}$

$$\operatorname{rref}\left(\left[\begin{array}{ccc|c} -1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 9 \\ -2 & -1 & 2 & 4 \end{array}\right] \right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 19/10 \\ 0 & 1 & 0 & -13/10 \\ 0 & 0 & 1 & 13/4 \end{array}\right]$$

$\Rightarrow \vec{x}$ is in the span of $\vec{v_1}, \vec{v_2}, \vec{v_3}$.

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 19/10 \\ -13/10 \\ 13/4 \end{bmatrix}$$

Bailey – p.2
Arkell

(6) $A \sim B$?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

$a = b \qquad b = a$

$-c = d \qquad -d = c$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\boxed{A \sim B \text{ and } S = \begin{bmatrix} a & a \\ c & -c \end{bmatrix}}$$

Verify

$a = 1,\ c = 2$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \checkmark$$

(7)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

$a = a \qquad 2b = b,\ b = 0$

$c = 2c,\ c = 0 \qquad 2d = 2d$

$$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basis of $W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\dim(A) = 2$

(8) $\vec{v_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$

$$T(\vec{v_1}) = A\vec{v_1} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [T(\vec{v_1})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\vec{v_2}) = A\vec{v_2} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 0 \end{bmatrix} = 9\vec{v_2} \Rightarrow [T(\vec{v_2})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix}$$

$$T(\vec{v_3}) = A\vec{v_3} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ -18 \end{bmatrix} = 9\vec{v_3} \Rightarrow [T(\vec{v_3})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

⑨ $W = \{p(x) \in P_2 : p'(1) = p(2)\}$

$p(x) = a + bx + cx^2$ $\qquad p'(1) = p(2)$

$p'(x) = b + 2cx$ $\qquad b + 2c = a + 2b + 4c$

$\qquad -a - b - 2c = 0 \Rightarrow a + b + 2c = 0$

(i) Zero Element:

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ because $-0 - 0 - 2(0) = 0$.

(ii) $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \in W$ because $-(a_1 + a_2) - (b_1 + b_2) - 2(c_1 + c_2) = 0$

$\Rightarrow (-a_1 - b_1 - 2c_1) + (-a_2 - b_2 - 2c_2) = 0$

$\Rightarrow 0 + 0 = 0$

(iii) $k \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in W$ because $-ka - kb - 2kc = 0$

$\Rightarrow k(-a - b - 2c) = 0$

$\Rightarrow k(0) = 0$

Standard Basis of $P_2 = \{1, x, x^2\}$

$\mathcal{B}_1 = \{-1 + x, 2 - x^2\}$ $\qquad \dim(W) = 2$

$\mathcal{B}_2 = \{-8x + 2x^2, 3 - 3x\}$

⑩ $T(M) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} M$ from $\mathbb{R}^{2\times2}$ to $\mathbb{R}^{2\times2}$

ⓐ $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ $\Rightarrow B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 2 & 0 & -4 & 0 \\ 0 & 2 & 0 & -4 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -4 & 0 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & -4 \end{bmatrix} \Rightarrow \left[T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -4 \end{bmatrix}$ $\quad \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

ⓑ $\mathrm{im}(B) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}\right)$ $\qquad \vec{v_3} = -2\vec{v_1} \Rightarrow 2\vec{v_1} + \vec{v_3} = \vec{0}$

$\qquad \vec{v_4} = -2\vec{v_2} \Rightarrow 2\vec{v_2} + \vec{v_4} = \vec{0}$

$\ker(B) = \mathrm{span}\left(\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}\right)$

ⓒ $\mathrm{im}(T) = \mathrm{span}\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right)$ $\quad \ker(T) = \mathrm{span}\left(\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\right)$

ⓓ T is NOT isomorphism because $\dim(\mathrm{im}(T)) = 2$ and $\dim(\ker(T)) = 2$.

T is isomorphism if $\dim(\ker(T)) = 0$.

Bailey - p.3
Arkell

(11) (a) $T: P_2 \to P_2$ given by $T(f) = 2f - f'$

(i) $T(f+g) = 2(f+g) - (f+g)' = 2f + 2g - f' - g' = (2f - f') + (2g - g')$
$= T(f) + T(g)$

(ii) $T(kf) = 2kf - kf' = k(2f - f') = kT(f)$

(b) $T(1) = 2(1) - (1)' = 2 - 0 = 2 \Rightarrow [T(1)]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = 2x - (x)' = 2x - 1 \Rightarrow [T(x)]_{\mathfrak{B}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$T(x^2) = 2x^2 - (x^2)' = 2x^2 - 2x \Rightarrow [T(x^2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad \text{rref}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) T is isomorphism because $\text{rref}(B) = I_3$ and

$\text{im}(B) = \text{span}\left(\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}\right)$

$\ker(B) = \{\vec{0}\}$

(12) (a) F; image is associated with $\mathbb{R}^4$, while kernel is associated with $\mathbb{R}^3$.

(b) T; when vectors are LD, they can be written in terms of another vector. This occurs with a non-zero vector.

Ex. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v_2} = 2\vec{v_1}$

(c) T; Standard basis of $P_2 = \{1, x, x^2\} \Rightarrow \dim(P_2) = 3$

(d) F; only 4 out of the 5 vectors could be linearly independent.

(e) T

(f) T; The image would be in $\mathbb{R}^4$.

(g) T; $2 + 3 = 5$

(h) T; $2 \times 3 = 6$

(i) F; (i) $T(f+g) = 3(f+g)(f+g)' = 3f^2 + fg' + gf' + (g^2)'$

(ii) $T(kf) = 3(kf)(kf') = k(3ff') = kT(f)$

(j) T : $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \Rightarrow \dim = 3$

Exam 3 MTH 331 Fall 2018 Total Pts:100 11/15/2018

Name:________________________ Total Received:

Show all work for full credit. Do not use calculator to find determinants. Extra 5 points included.

1. Find the angle between the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. (4 Pts)

2. For the subspace $W = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)$ of $\mathbb{R}^4$, find a basis for $W^\perp$ and then find an *orthonormal* basis for $W^\perp$. (8 Pts)

3. Perform the Gram-Schmidt process to find the QR factorization of the matrix
$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}. \qquad \text{(10 Pts)}$$

4. Consider the subspace W of $\mathbb{R}^4$ spanned by the vectors $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$.

   Find the matrix M of the orthogonal projection onto W. (4 Pts)

5. Consider the subspace $V = im(A)$ of $\mathbb{R}^4$, where $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$.

   Find orthogonal projection, $proj_V(\vec{x})$, for $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$. (6 Pts)

6. Use Sarrus's rule to find the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix}$. (5 Pts)

7. Using Gaussian elimination, turn into upper triangular to find the determinant of A for
$$A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}. \qquad \text{(5 Pts)}$$

8. Use Gaussian elimination to find the determinant of the matrix
$$A = \begin{bmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}. \qquad \text{(7 Pts)}$$

9. Find the eigenvalues of the matrices $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. (10 Pts)

10. Find the determinant of the transformation $T(f(t)) = 2f - f'$ from P_2 to P_2.
Is the linear transformation T invertible? (6 Pts)

11. Use Cramer's rule to solve the system (you can use calculator for the determinants)

$$\begin{aligned} x + 2y + z &= 5 \\ 2x + 2y + z &= 6 \\ x + 2y + 3z &= 9. \end{aligned} \qquad \text{(6 Pts)}$$

12. Find the area of the 2-parallelepiped defined by the vectors $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (5 Pts)

13. Find the classical adjoint and determinant of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.
Use them to find the inverse $A^{-1} \left(= \frac{adj(A)}{det(A)} \right)$ of the matrix A. (8 Pts)

14. Consider a 4×4 matrix A with rows $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}$. If $det(A) = 4$, find the determinants of
(a) $det \begin{bmatrix} \overrightarrow{v_1} \\ 2\overrightarrow{v_2} \\ 3\overrightarrow{v_3} \\ \overrightarrow{v_4} \end{bmatrix}$ (b) $det \begin{bmatrix} 2\overrightarrow{v_2} \\ \overrightarrow{v_1} \\ \overrightarrow{v_3} \\ \overrightarrow{v_4} \end{bmatrix}$ (c) $det \begin{bmatrix} \overrightarrow{v_1} \\ \overrightarrow{v_1} + 2\overrightarrow{v_2} \\ \overrightarrow{v_2} + 3\overrightarrow{v_3} \\ \overrightarrow{v_1} + \overrightarrow{v_2} + \overrightarrow{v_4} \end{bmatrix}$. (6 Pts)

15. State whether the following statements are true or false. (15 Pts)
(a) If $det(A) = 10$, then 0 cannot be an eigenvalue of the matrix A.
(b) If A is an $n \times n$ matrix such that $AA^T = I_n$, then A must be an orthogonal matrix.
(c) If A and B are symmetric $n \times n$ matrices, then AB must be symmetric as well.
(d) The equation $det(A^T) = det(A)$ holds for all $n \times n$ matrices.
(e) If A and B are orthogonal 2×2 matrices, then $AB = BA$.
(f) $\|\overrightarrow{x} + \overrightarrow{y}\|^2 = \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2$ must hold for any orthogonal vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ in $\mathbb{R}^n$.
(g) If all entries of a 7×7 matrices are 7, then $det(A) = 7^7$.
(h) If $A = [\overrightarrow{u}\,\overrightarrow{v}\,\overrightarrow{w}]$ is any 3×3 matrix, then $det(A) = \overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w})$.
(i) The determinant of any $n \times n$ matrix is the product of its diagonal entries.
(j) There exists a real 5×5 matrix without any real eigenvalues.
(k) The equation $det(4A) = 4det(A)$ holds for all 4×4 matrices A.
(l) The equation $det(-A) = det(A)$ holds for all 4×4 matrices.
(m) The eigenvalues of any triangular matrix are its diagonal entries.
(n) If $\overrightarrow{v}$ is an eigenvector of A, then $\overrightarrow{v}$ must be an eigenvector of A^3 as well.
(o) The $det(A)$ is the product of its eigenvalues, counted with their algebraic multiplicities.

Chloe Marcum

1. $\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos\theta \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \left(\sqrt{1^2+(-1)^2+1^2}\right)\left(\sqrt{1^2+2^2+1^2}\right) \cos\theta$$

$$0 = (\sqrt{3})(\sqrt{6}) \cos\theta$$

$$\cos\theta = 0$$

$$\boxed{\theta = \pi/2}$$

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2. $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1 + x_2 + x_3 + x_4 = 0 \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = x_1 - x_2 - x_3 + x_4 = 0$$

$$\left| \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 - x_3 + x_4 = 0 \end{array} \right| \quad \operatorname{rref} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$\vec{v}_1 = \vec{v}_4 \Rightarrow \vec{v}_1 - \vec{v}_4 = 0$

$\vec{v}_2 = \vec{v}_3 \Rightarrow \vec{v}_2 - \vec{v}_3 = 0$

$$\boxed{\text{A basis for } W^\perp = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{u}_2 : \ \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \cancel{\left(\frac{1}{\sqrt{2}}(0) \right)}^{0} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{An orthonormal basis of } W^\perp = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\}}$$

(3.) $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ $\boxed{A = QR = \underset{A}{\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}} = \underset{Q}{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}} \underset{R}{\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}}}$

$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\underset{r_{11}}{2}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\vec{u}_2$: $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \left(\underbrace{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}}_{= 4 = r_{12}}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$\overset{8}{1/2(3+1+1+3)}$

$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\underset{r_{22}}{2}}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$\vec{u}_3$: $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \left(\overbrace{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}}^{\frac{1}{2}(10) = 5 = r_{13}}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\overbrace{\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}}^{\frac{1}{2}(6) = 3 = r_{23}}\right)\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{\underset{r_{33}}{2}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ $= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

(4.) $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ $M = QQ^T$

$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $Q^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

(work shown above)

$M = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}$

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(5) $V = \text{im}(A)$

$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$ Since the vectors of A are LI, the $\text{im}(A) = A$.

$\vec{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{u}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

(work shown in #3)

$$\text{proj}_V(\vec{x}) = (\vec{u}_1 \cdot \vec{x}_1)\vec{u}_1 + (\vec{u}_2 \cdot \vec{x}_1)\vec{u}_2$$

$$= \left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 6 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{2}\overset{6}{(12)}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \left(\overset{1}{\frac{1}{2}(2)}\right)\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 5/2 \\ 5/2 \\ 7/2 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} 7 \\ 5 \\ 5 \\ 7 \end{bmatrix}}$$

(6) $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 3 & 4 \\ 2 & 1 & 5 \end{bmatrix}$

$$\begin{matrix} 1 & 2 & -1 & 1 & 2 \\ -2 & 3 & 4 & -2 & 3 \\ 2 & 1 & 5 & 2 & 1 \\ - & - & - & + & + & + \end{matrix}$$

$15 + 16 + 2 + 6 - 4 + 20$

$\boxed{\det(A) = 55}$

(7) $A = \begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}$

$\boxed{\det(A) = -3}$

$$\begin{vmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{vmatrix} \begin{matrix} \\ +R_1 \\ +2R_1 \end{matrix} \rightarrow \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix} \begin{matrix} \\ \\ -2R_2 \end{matrix} \rightarrow \begin{vmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$

$(-3)(1)(1) = -3$

(8)
$$\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \begin{matrix} -3R_3 \\ -5R_3 \\ \\ \\ \end{matrix} \longrightarrow \begin{vmatrix} 10 & -4 & 0 & 10 \\ 18 & -9 & 0 & 20 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \rightarrow (1)\begin{vmatrix} 10 & -4 & 10 \\ 18 & -9 & 20 \\ 1 & -4 & 6 \end{vmatrix} \begin{matrix} -10R_3 \\ -18R_3 \\ \\ \end{matrix} \longrightarrow \begin{vmatrix} 0 & 36 & -50 \\ 0 & 63 & -88 \\ 1 & -4 & 6 \end{vmatrix} \rightarrow$$

$$(1)(1)\begin{vmatrix} 36 & -50 \\ 63 & -88 \end{vmatrix} = (-3168 + 3150) = \boxed{-18 = \det(A)}$$

(9) $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

$\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6 = 0$

$-6 - 3\lambda + 2\lambda + \lambda^2 - 6 = 0$

$\lambda^2 - \lambda - 12 = 0$

$(\lambda - 4)(\lambda + 3) = 0$

$\boxed{\text{For } A,\ \lambda_1 = 4 \quad \lambda_2 = -3}$

$B = \begin{bmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ $\det(B - \lambda I_3) = 0$

$\begin{vmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix}$

$(4-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - 1\begin{vmatrix} -5 & 1 \\ 1 & -1-\lambda \end{vmatrix} + 0$

$(4-\lambda)\left(-\lambda(-1-\lambda) + 1\right) - \left(-5(-1-\lambda) - 1\right) = 0$

$(4-\lambda)(\lambda + \lambda^2 + 1) - (5 + 5\lambda - 1) = 0$

$4\lambda + 4\lambda^2 + 4 - \lambda^2 - \lambda^3 - \lambda - 5 - 5\lambda + 1 = 0$

$-2\lambda + 3\lambda^2 - \lambda^3 = 0$

$-\lambda^3 + 3\lambda^2 - 2\lambda = 0$

$\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\lambda(\lambda^2 - 3\lambda + 2) = 0$

$\lambda(\lambda - 2)(\lambda - 1) = 0$

$\boxed{\text{For } B,\ \lambda = 0,\ \lambda = 2,\ \lambda = 1}$

(10) $b + 2cx$

$$\begin{array}{ccc} f(t) = a + bx + cx^2 & \longrightarrow & T(f(t)) = 2a + 2bx + 2cx^2 - b - 2cx \\ & & = (2a - b) + (2bx - 2cx) + 2cx^2 \\ \downarrow & & \downarrow \\ [f(t)]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \longrightarrow & \begin{bmatrix} 2a - b \\ 2b - 2c \\ 2c \end{bmatrix} = [T(f(t))]_B \end{array}$$

$B = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(B) = 2(2)(2) = \boxed{8} \Rightarrow$ T is invertible.

(11) $x = \dfrac{\begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \dfrac{-4}{-4} = 1$ $\quad y = \dfrac{\begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \dfrac{-4}{-4} = 1$ $\quad z = \dfrac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \dfrac{-8}{-4} = 2$

$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}$

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(12) Area $= \sqrt{\det(A^TA)} = \boxed{\sqrt{6}}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$\det(A^TA) = (42 - 36) = 6$

(13) $\operatorname{adj}(A) = \begin{bmatrix} +|A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T = \begin{bmatrix} -2 & 3 & 5 \\ 14 & -7 & -7 \\ 4 & 1 & -3 \end{bmatrix}^T = \boxed{\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}}$

$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$ $\det(A)$: $\begin{matrix} 2 & 1 & 3 & 2 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 4 & -2 & 1 & 4 \\ - & - & - & + & + & + \end{matrix}$ $4 + 1 + 12 + 3 - 8 + 2 = \boxed{14 = \det(A)}$

$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \boxed{\frac{1}{14}\begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}}$

(14) a.) $\det\left(\begin{bmatrix} \vec{v}_1 \\ 2\vec{v}_2 \\ 3\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}\right) = \boxed{24}$

pull 2 and 3 out
$6 \cdot 4 = 24$

b.) $\det\begin{bmatrix} 2\vec{v}_2 \\ \vec{v}_1 \\ \vec{v}_3 \\ \vec{v}_4 \end{bmatrix} = \boxed{-8}$

swap 2 rows (−)
pull out $2 \to 4 \cdot 2 = 8$

c.) $\det\begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + 2\vec{v}_2 \\ \vec{v}_2 + 3\vec{v}_3 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_4 \end{bmatrix} = \boxed{24}$

adding rows doesn't change det. pull out 2 from $2\vec{v}_2$ and 3 from $3\vec{v}_3$ to get $24 = (6 \cdot 4)$

(15) a.) $\boxed{\text{true}}$

b.) $\boxed{\text{true}}$

c) $\boxed{\text{false}}$

d) $\boxed{\text{true}}$

e) $\boxed{\text{false}}$

f) $\boxed{\text{true}}$

g) $\boxed{\text{false}}$ det $= 0$

h) $\boxed{\text{true}}$

i) $\boxed{\text{false}}$

j) $\boxed{\text{false}}$

k) $\boxed{\text{false}}$

l) $\boxed{\text{true}}$ $(-1)^4 = 1$

m) $\boxed{\text{true}}$

n) $\boxed{\text{true}}$

o) $\boxed{\text{true}}$

c) $A = A^T$ $B = B^T$

$(AB)^T = B^T A^T = BA$

Final Exam MTH 331 Fall 2018 Total Pts:100 12/6/18

Name:__________________________ Received:

Show all work for full credit. Write all your solutions on the given blank papers

Points distribution:

Problem No.	1	2	3	4	5	6	7	8	9	10	Total
Total Points	8	10	8	10	8	8	10	8	10	20	100

1. (**No Calculator**) Use Gauss-Jordan elimination to solve the following system
$2x + y - z = 3, \quad x + y + z = 4, \quad x + 3y + 3z = 10.$

2. Find the redundant column vectors of the matrix A, and then find a basis of the image of A and a basis of the kernel of A, where
$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}.$$

3. Let P_2 be the linear space of all polynomials of degree two or less. Prove that the transformation $T : P_2 \to P_2$ defined by $T(f(x)) = f(x) - 2f''(x)$ is linear. Find the $\mathcal{B}$-matrix B of the transformation T and use it to find kernel of T. Show that T is isomorphism.

4. (**No Calculator**) Find the QR-factorizations of the matrix
$$A = \begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}.$$

5. Let V be the plane in $\mathbb{R}^3$ that is spanned by the vectors $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

   Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto the plane V.

6. (**No Calculator**) Use the Gaussian elimination and Laplace expansion to find the determinant of the following matrix.
$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

7. (**No Calculator**) For the matrix A, find all (real) eigenvalues, then find a basis of each eigenspace, and find an eigenbasis, if it exists, where the matrix A is
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Is the matrix A diagonalizable? If so, find the matrices S and D which diagonalize the matrix A.

8. Find all the eigenvalues of the linear transformation $T(f(x)) = f(2x-3)$ from P_2 to P_2. Is T diagonolizable ? If it is, then find the diagonal matrix D.

9. Find an orthonormal eigenbasis of the symmetric matrix $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$.

10. State True or False. Give short reasons if possible.
(a) A system of four linear equations in four unknowns can be inconsistent.
(b) If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$ and rank of the matrix $A = [\vec{u}\,\vec{v}\,\vec{w}]$ is 2, then $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.
(c) If W_1 and W_2 are subspaces of a linear space V, then the intersection $W_1 \cap W_2$ must be a subspace of V as well.
(d) The formula $AB = BA$ holds for all $n \times n$ invertible matrices A and B.
(e) The function $T(f) = 3f - 4f'$ from C^∞ to C^∞ is a linear transformation.
(f) The polynomials of degree less than 3 form a 3-dimensional subspace of the linear space of all polynomials.
(g) If T is an isomorphism, then T^{-1} must be an isomorphism as well.
(h) All linear transformations from P_3 to $\mathbb{R}^{3\times 3}$ are isomorphism.
(i) The equation $\det(A^T) = \det(A)$ holds for all invertible matrices.
(j) The property $A(B+C) = AB + AC$ holds if the products make sense.
(k) If the determinant of a square matrix is 1 or -1, then A must be an orthogonal matrix.
(l) If an $n \times n$ matrix A is diagonalizable, then there must be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
(m) All diagonalizable matrices are invertible.
(n) The equation $\det(A+B) = \det(A) + \det(B)$ holds for all 3×3 matrices A and B.
(o) If two 3×3 matrices A and B both have the eigenvalues 1, 2 and 3, then A must be similar to B.
(p) There exists a subspace V of $\mathbb{R}^7$ such that $dim(V) = dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.
(q) If the determinant of a 4×4 matrix A is 4, then its rank must be 4.
(r) The eigenvalues of any triangular matrix are its diagonal entries.
(s) If 1 is the only eigenvalue of an $n \times n$ matrix A, then A must be an I_n.
(t) If A is diagonalizable 4×4 matrix with $A^4 = 0$, then A must be a zero matrix.

Skylaar Measc
12/6/2018

1. $2x + y - z = 3$
$x + y + z = 4$
$x + 3y + 3z = 10$

The associated coefficient matrix

$$A = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \end{array}\right].$$

$$\operatorname{rref}(A) = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 1 & -1 & 3 \\ 1 & 3 & 3 & 10 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -3 & -5 \\ 0 & 2 & 2 & 6 \end{array}\right] \begin{array}{l} \\ (-1)R_2 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 2 & 6 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -4 & -4 \end{array}\right] \begin{array}{l} \\ \\ (-\tfrac{1}{4})R_3 \end{array} = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - 3R_3 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, $x = 1$, $y = 2$, and $z = 1$.

2. $$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix} \quad \operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{array}$$

$2\vec{v}_1 = \vec{v}_2$, $3\vec{v}_1 - 4\vec{v}_3 = \vec{v}_4$, and $-4\vec{v}_1 + 5\vec{v}_3 = \vec{v}_5$

Thus, $\vec{v}_2$, $\vec{v}_4$, and $\vec{v}_5$ are redundant.

A basis of $\operatorname{im}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix} \right\}$

A basis of $\ker(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 5 \\ 0 \\ -1 \end{bmatrix} \right\}$

3. Let P_2 be the linear space of all polynomials of degree two or less.

Proof: We will show that the transformation

$T: P_2 \to P_2$, $T(f(x)) = f(x) - 2f''(x)$ is linear by showing it contans the zero element, case ①, it is closed under scalar multiplication, case ②, and it is closed under addition.

For case ①, let $f(x) = 0$. Then,

$T(f(x)) = 0 - 2(0) = 0 - 0 = 0$. Thus, T contains the zero element.

For case ②, let $f(x) \in P_2$ and $g(x) \in P_2$. Then,

$$\begin{aligned} T(f+g) &= (f+g) - 2(f+g)'' \\ &= f + g - 2f'' - 2g'' \\ &= f - 2f'' + g - 2g'' \\ &= T(f) + T(g). \end{aligned}$$

Hence, T is closed under addition.

For case ③, let $f(x) \in P_2$ and $k \in \mathbb{R}$. Then,

$$\begin{aligned} T(kf) &= (kf) - 2(kf)'' \\ &= kf - 2kf'' \\ &= k(f - 2f'') \\ &= kT(f). \end{aligned}$$

Hence, T is closed under scalar multiplication.

Therefore, T is linear. //

$$B: \{1, x, x^2\} \xrightarrow{T} (a+bx+cx^2) - 2(a+bx+cx^2)''$$

$$= a + bx + cx^2 - 2(b + 2cx)'$$

$$= a + bx + cx^2 - 2(2c)$$

$$= (a - 4c) + bx + cx^2$$

$$\downarrow \qquad\qquad \downarrow$$

$$[x]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad [T]_B = \begin{bmatrix} a - 4c \\ b \\ c \end{bmatrix}$$

$$B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}$$

Since, $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent, the basis of $\ker(B) = \{\vec{0}\}$. This implies the basis of $\ker(T) = \{0\}$. Therefore, T is isomorphic.

4.
$$\underset{M}{\begin{bmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}} = \underset{Q}{\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}} \underset{R}{\begin{bmatrix} 3 & 0 & 12 \\ 0 & 3 & -12 \\ 0 & 0 & 6 \end{bmatrix}}$$

$$\|\vec{v}_1\| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3 \; r_{11}$$

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{u}_3 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - \left[\underset{r_{12}}{0} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - 0 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{(-2)^2 + (1)^2 + (2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3 \, r_{22}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \left[(\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 + (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 \right]$$

$$= \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} - \left[12 \, r_{13} \cdot \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + -12 \, r_{23} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right]$$

$$\|\vec{v}_3^{\perp}\| = \sqrt{(2)^2 + (-4)^2 + (4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6 \, r_{33}$$

$$= \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \\ -8 \end{bmatrix} \right] = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ +4 \end{bmatrix}$$

5. $\vec{v_1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ V a plane in $\mathbb{R}^3$ | $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$.

$\vec{v_1}$ and $\vec{v_2}$ are already perpendicular because

$\vec{v_1} \cdot \vec{v_2} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = 0$. Thus,

$\vec{u_1} = \dfrac{\vec{v_1}}{\|\vec{v_1}\|} = \dfrac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{u_2} = \dfrac{\vec{v_2}}{\|\vec{v_2}\|} = \dfrac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

Thus, M, the matrix of projection of V is

$M = QQ^T$ where $Q = [\vec{u_1}\ \vec{u_2}]$, so

$M = \frac{1}{3} \cdot \frac{1}{3} \begin{bmatrix} 2 & -2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then

$$\text{Proj}_V(\vec{x}) = M\vec{x} = \frac{1}{9} \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 8 \\ 11 \\ 7 \end{bmatrix}.$$

6. $$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 0 & 7 & 5 & 3 \\ 1 & 1 & 1 & 2 \end{vmatrix} \begin{matrix} \\ R_2 - R_1 \\ \\ R_2 - R_1 \end{matrix} = \underset{\uparrow \text{ Laplace CI}}{\overset{\text{Laplace rI} \rightarrow}{\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 7 & 5 & 3 \\ 0 & 0 & -1 & 1 \end{vmatrix}}} = (1) \underset{\uparrow \text{ Laplace CI}}{\overset{\text{Laplace rII} \rightarrow}{\begin{vmatrix} 0 & 0 & -2 \\ 7 & 5 & 3 \\ 0 & -1 & 1 \end{vmatrix}}} = (-7) \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}$$

$= (-7)(0 - 2) = 14$

7.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ -5 & -\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$$

Laplace r II

Laplace c II

$= (-\lambda)(1-\lambda)(1-\lambda) = 0$

$\lambda = 0$, $\lambda = 1$

alg mult of 1 alg mult of 2

$$E_0 = \ker(A - 0 I_3) = \ker\left(\begin{array}{ccc} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

A basis of E_0

$\vec{v}_2 = 0$

$\lambda = 0$ has geo mult of 1

$$E_1 = \ker(A - 1 I_3) = \ker\left(\begin{array}{ccc} \begin{bmatrix} 0 & 0 & 0 \\ -5 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \end{array}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$$

A basis of E_1

$\vec{v}_1 = 5\vec{v}_2$, $\vec{v}_3 = -2\vec{v}_2$

$\vec{v}_1 - 5\vec{v}_2 = \vec{0}$, $2\vec{v}_2 + \vec{v}_3 = \vec{0}$

$\lambda = 1$ has geo mult of 2

Since, alg multiplicity = geo multiplicity for all λ,

an eigenbasis of A is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$.

$S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -5 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$8.\ \mathcal{B}: \{1, x, x^2\} \xrightarrow{T} a + b(2x-3) + c(2x-3)^2$$

$$= a + 2bx - 3b + c(4x^2 - 12x + 9)$$

$$= a + 2bx - 3b + 4cx^2 - 12cx + 9c$$

$$= (a - 3b + 9c) + (2b - 12c)x + 4cx^2$$

$$[x]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad [T]_{\mathcal{B}} = \begin{bmatrix} a - 3b + 9c \\ 2b - 12c \\ 4c \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 9 \\ 0 & 2 & -12 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda = 1, \lambda = 2, \lambda = 4$

$$E_1 = \ker(A - 1I_3) = \ker\left(\begin{array}{c} \begin{bmatrix} 0 & -3 & 9 \\ 0 & 1 & -12 \\ 0 & 0 & 3 \end{bmatrix} \\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3 \end{array}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \rightarrow \text{eigenfunction: } 1$$

$\lambda = 1$ has geo mult of 1 $\qquad \vec{v}_1 = 0$

$$E_2 = \ker(A - 2I_3) = \ker\left(\begin{array}{c} \begin{bmatrix} -1 & -3 & 9 \\ 0 & 0 & -12 \\ 0 & 0 & 2 \end{bmatrix} \\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3 \end{array}\right) = \text{span}\left(\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}\right) \rightarrow \text{eigenfunction } (3-x)$$

$\lambda = 2$ has geo mult of 1 $\qquad 3\vec{v}_1 = \vec{v}_2$

$$E_4 = \ker(A - 4I_3) = \ker\left(\begin{array}{cc} \begin{array}{ccc} 36 & -18 & \\ 0 & -12 & \\ 0 & & \end{array} \\ \begin{bmatrix} -3 & -3 & 9 \\ 0 & -2 & -12 \\ 0 & 0 & 0 \end{bmatrix} \end{array}\right) = \text{span}\left(\begin{bmatrix} 9 \\ -6 \\ 1 \end{bmatrix}\right) \quad \text{eigenfunction } 9 - 6x + x^2 = (3-x)^2$$

$\lambda = 4$ has geo mult of 1

$$\ker\left(\begin{array}{c} \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \\ \vec{v}_1\ \vec{v}_2\ \vec{v}_3 \end{array}\right) \quad \vec{v}_3 = -9\vec{v}_1 + 6\vec{v}_2$$

T is diagonalizable because alg mult = geo mult for a λ, and

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Eigenbasis of T is $\{1, (3-x), (3-x)^2\}$

9. $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

$$\det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & 1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} \leftarrow \text{Laplace r II} = (-2)\begin{vmatrix} 2 & 0 \\ 2 & -1-\lambda \end{vmatrix} + (1-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$$

↑ Laplace c II

$$= (-2)(-2-2\lambda) + (1-\lambda)(+\lambda + \lambda^2 - 4) = \cancel{4} + \underline{4\lambda} + \underline{\lambda} \cancel{+\lambda^2} \cancel{-4} \cancel{-\lambda^2} - \lambda^3 + \underline{4\lambda}$$

$$= -\lambda^3 + 9\lambda = -\lambda(\lambda^2 - 9) = -\lambda(\lambda+3)(\lambda-3) = 0$$

$\lambda = 0, \lambda = 3, \lambda = -3$

all with alg mult 1

$E_0 = \ker(A) = \ker\left(\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\right)$

$\lambda = 0$ has geo mult 1

$\downarrow$

$\ker\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$2\vec{v}_3 = -\vec{v}_1 + 2\vec{v}_2$

$E_3 = \ker(A - 3I_3) = \ker\left(\begin{bmatrix} -3 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -4 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right)$

$\lambda = 3$ has geo mult 1 $\downarrow$

$\ker\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$\vec{v}_3 = -2\vec{v}_1 - 2\vec{v}_2$

$E_{-3} = \ker(A + 3I_3) = \ker\left(\begin{bmatrix} 3 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}\right)$

$\lambda = -3$ has geo mult 1 $\quad \ker\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}\right)$

$2\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$

Since, alg mult = geo mut for all λ, A has an eigenbasis and since the vectors that span it are perpendicular,

an orthonomal eigenbasis of A: $\left\{ \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \right\}$.

10. a) True, if any are LD this could occur

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & d \end{array}\right]$$ where a, b, c any $\in \mathbb{R}$ and $d \neq 0$

b) True $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$ a, b any $\in \mathbb{R}$

c) True, contains 0 element and closed, only more restricted

d) False, only if $B = A^{-1}$ and $A = B^{-1}$

e) True, contains 0 function, + closed under linear combination

f) True, $\mathcal{B} = \{1, x, x^2\}$ dim $= 3$

g) True, ker (T^{-1}) should be empty and im(T^{-1}) should be full if invertible

h) False, cannot know without checking

i) True, holds for all matrices, full stop

j) , I don't believe distributive property is ever defined for matrix multiplication

True → if the product make sense

k) False, converse is true

l) True, definition of diagonalizable

m) False, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

det = 0 but diagonal

det = 0

n) False

o) True,

p) False, $\dim(V) + \dim(V^{\perp}) = 7$

q) True, if $\det(A) \neq 0$ then LI and invertible, so

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ so rank(A) = 4

r) True, $\begin{vmatrix} a-\lambda & b & c \\ 0 & d-\lambda & e \\ 0 & 0 & f-\lambda \end{vmatrix} = (a-\lambda)\begin{vmatrix} d-\lambda & e \\ 0 & f-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda)(f-\lambda) = 0$

$\lambda = a, d, f$

s) False, eg. $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

t) True