

Name: \_\_\_\_\_

Received:

*Show all work for full credit. Write all your solutions on the blank papers.*

1. Verify that the function  $y = xe^{2x}$  is a solution of the differential equation  $y'' - 4y' + 4y = 0$ . (5 Pts)
2. If  $y = c_1e^{2x} + c_2xe^{2x}$  is a two-parameter family of solutions of the differential equation  $y'' - 4y' + 4y = 0$ , find a solution of the IVP consisting of the differential equation and the initial conditions  $y(0) = -1$ ,  $y'(0) = 2$ . (6 Pts)
3. For the autonomous first-order differential equation  $\frac{dy}{dx} = y^2 - y$ ,
  - (a) find the critical points and phase portrait,
  - (b) classify each critical point as asymptotically stable, unstable, or semi-stable,
  - (c) by hand, sketch typical solution curves in the region in the  $xy$ -plane determined by the graphs of the equilibrium solutions. ( 6 Pts)
4. Classify each of the following differential equations as separable equations, linear equations, exact equations, homogeneous equations and Bernoulli's equations, and then solve any **Seven** of them. ( $7 \times 9 = 63$  Pts)
  - (i)  $x\frac{dy}{dx} = y^2 - y$ ,  $y(0) = 1$
  - (ii)  $(x^2 + xy + y^2)dx - x^2dy = 0$
  - (iii)  $\frac{dy}{dx} + y^2 \sin x = 0$
  - (iv)  $x\frac{dy}{dx} + y = x^2y^2$
  - (v)  $\cos x\frac{dy}{dx} + (\sin x)y = 1$
  - (vi)  $(y^3 - x - y^2 \sin x)dx + (3xy^2 + 2y \cos x)dy = 0$
  - (vii)  $x\frac{dy}{dx} + y = e^x$ ,  $y(1) = 2$
  - (viii)  $(x + y)^2dx + (2xy + x^2 - 1)dy = 0$
  - (ix)  $\frac{dy}{dx} = (x + y + 2)^2$
5. (Any Two) Solve the following problems. ( $2 \times 10 = 20$  Pts)
  - (a) The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time  $t$ . After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?
  - (b) A thermometer is removed from a room where the temperature is  $70^\circ$  F and is taken outside, where the air temperature is  $20^\circ$  F. After one minute the thermometer reads  $50^\circ$  F. What is the reading of the thermometer at  $t = 3$  min? How long will it take for the thermometer to reach  $25^\circ$  F?
  - (c) A large tank is filled to capacity with 400 gallons of pure water. Brine containing 3 pounds of salt per gallon is pumped into the tank at the rate of 4 gal/min. The well-mixed solution is pumped out at the same rate. If 100 pounds of salt were dissolved initially in the 400 gallons, find the number  $A(t)$  of pounds of salt in the tank at time  $t$ .

$$1. \quad y = xe^{2x}$$

$$y' = x(2e^{2x}) + e^{2x}$$

$$y'' = x(4e^{2x}) + 2e^{2x} + 2e^{2x}$$

$$y'' - 4y' + 4y = 0$$

$$4xe^{2x} + 2e^{2x} + 2e^{2x} - 4(2xe^{2x} + e^{2x}) + 4(xe^{2x}) = 0$$

$$\cancel{4xe^{2x}} + \cancel{4e^{2x}} - 8xe^{2x} - \cancel{4e^{2x}} + \cancel{4xe^{2x}} = 0 \quad \checkmark$$

$$2. \quad y = C_1 e^{2x} + C_2 x e^{2x}, \quad y(0) = -1, \quad y'(0) = 2$$

$$-1 = C_1 e^{2(0)} + C_2 (0) e^{2(0)} \quad y' = -2e^{2x} + C_2 [x(2e^{2x}) + e^{2x}]$$

$$-1 = C_1$$

$$2 = -2e^{2(0)} + C_2 [(0)(2e^{2(0)}) + e^{2(0)}]$$

$$y = -e^{2x} + C_2 x e^{2x}$$

$$2 = -2 + C_2 \Rightarrow C_2 = 4$$

$$\boxed{y = -e^{2x} + 4xe^{2x}}$$

3.

$$a) \frac{dy}{dx} = y^2 - y = y(y-1) = f(y)$$

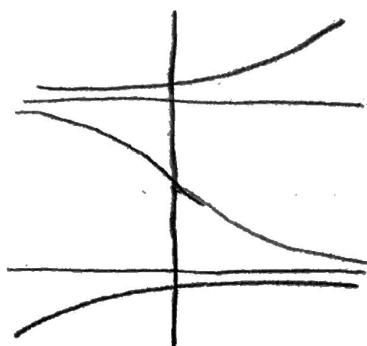
b) CP = 0: asymptotically stable

CP = 1: unstable

$$f(y) = 0 \text{ when } y=0 \text{ & } y=1$$



c)



i) separable:

$$x \frac{dy}{dx} = y^2 - y, \quad y(0) = 1$$

$$\frac{dy}{dx} = \frac{(y^2 - y)}{x}$$

$$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x}$$

$$-\int \frac{dy}{y} + \int \frac{dy}{y-1} = \int \frac{dx}{x}$$

$$-\ln|y| + \ln|y-1| = \ln|x| + C_1$$

$$\ln\left(\frac{|y-1|}{|y|}\right) = \ln|x| + C_1$$

$$\frac{y-1}{y} = e^{\ln|x| + C_1} = e^{C_1} \cdot e^{\ln|x|}$$

$$\left[ \frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1} \right] y(y-1)$$

$$1 = A(y-1) + By$$

$$y=0 \div 1 = -A \rightarrow A = -1$$

$$y=1 \div 1 = B$$

$$\frac{y-1}{y} = CX$$

$$\rightarrow \frac{1-1}{1} = C(0) = 0$$

$$\text{if } C=0, \text{ then } \frac{y-1}{y} = 0$$

$$1 - \frac{1}{y} = 0$$

$$1 = \frac{1}{y} \Rightarrow \boxed{y=1} \quad 1 - \frac{1}{y} = CX$$

$$\boxed{\frac{y-1}{y} = CX \text{ for any } C}$$

ii) homogeneous

$$(x^2 + xy + y^2)dx - x^2 dy = 0 \quad \begin{aligned} y &= ux \\ dy &= udx + xdu \end{aligned}$$

$$\begin{aligned} My &= x+2y & \frac{My-Nx}{N} &= \frac{x+2y-(-2x)}{-x^2} = \frac{3x+2y}{-x^2} & \times \\ Nx &= -2x \end{aligned}$$

$$(x^2 + x^2u + u^2x^2)dx - x^2(u dx + x du) = 0$$

$$(x^2 + x^2u + u^2x^2)dx - x^2u dx - x^3 du = 0$$

$$(x^2 + u^2x^2)dx - x^3 du = 0$$

$$\left[ x^2(1+u^2)dx - x^3 du = 0 \right] \frac{1}{x^2}$$

$$(1+u^2)dx - x du = 0$$

$$(1+u^2)dx = x du$$

$$\left\{ \frac{dx}{x} = \int \frac{du}{1+u^2} \right.$$

$$\ln|x| = \tan^{-1}(u) + C$$

$$\boxed{\ln|x| = \tan^{-1}\left(\frac{y}{x}\right) + C}$$

iii) separable:

$$\frac{dy}{dx} + y^2 \sin x = 0$$

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\int y^{-2} dy = \int -\sin x dx$$

$$-y^{-1} = \cos x + C,$$

$$\frac{1}{y} = -\cos x + C$$

$$y = \frac{1}{C - \cos x}$$

Hart, Madeline  
IF:  $e^{\int \tan x dx}$   
 $= e^{\ln |\sec x|}$

$$\checkmark v) \cos x \frac{dy}{dx} + (\sin x)y = 1$$

$$\frac{dy}{dx} + (\tan x)y = \sec x \quad = \sec x$$

$$\int d[\sec x(u)] = \int \sec^2 x dx$$

$$(\sec x)y = \tan x + C$$

$$y = \left[ \frac{\sin x}{\cos x} + C \right] \cos x$$

$$y = \sin x + (\cos x)C$$

iv) bernoulli

$$\left[ x \frac{dy}{dx} + y = x^2 y^2 \right] \frac{1}{x}$$

$$\frac{dy}{dx} + x^{-1}y = x y^2$$

\* DID NOT SOLVE \*

$$u = y^{1-n}$$

$$u = y^{1-2} = y^{-1} = \frac{1}{y} \rightarrow u^2 = y^{-2}, y = \frac{1}{u}$$

$$y = \frac{1}{u}$$

$$\frac{du}{dx} = -y^{-2} \frac{dy}{dx} = -u^2 \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{y^2} \frac{du}{dx} = -\frac{1}{u^2} \frac{du}{dx}$$

$$\left[ -\frac{1}{u^2} \frac{du}{dx} + x^{-1} \frac{1}{u} = x \frac{1}{u^2} \right] (-u^2)$$

$$\frac{du}{dx} - \frac{1}{x}u = -x \quad (\text{linear})$$

$$\text{IF} = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = e^{\ln|\frac{1}{x}|} = \frac{1}{x}$$

$$\int d\left(\frac{1}{x}u\right) = \int (-x)\left(\frac{1}{x}\right) dx$$

$$\begin{aligned} \frac{u}{x} &= -x + C \\ u &= -x^2 + cx \end{aligned} \rightarrow \frac{1}{y} = -x^2 + cx \rightarrow y = \frac{1}{-x^2 + cx}$$

vi) exact:

$$(y^3 - x - y^2 \sin x)dx + (3xy^2 + 2y \cos x)dy = 0$$

$$\begin{aligned} My &= 3y^2 - 2y \sin x \\ Nx &= 3y^2 - 2y \sin x \end{aligned}$$

$$f_x = y^3 - x - y^2 \sin x$$

↓ wrt x

$$\int (y^3 - x - y^2 \sin x)dx$$

$$f_y = 3xy^2 + 2y \cos x$$

↓ wrt y

$$\int (3xy^2 + 2y \cos x)dy$$

$$f(x, y) = y^3x - \frac{1}{2}x^2 + y^2 \cos x + g(y) \quad f(x, y) = xy^3 + y^2 \cos x + h(x)$$

$$\boxed{f(x, y) = xy^3 + y^2 \cos x - \frac{1}{2}x^2 = C}$$

vii) linear:

$$x \frac{dy}{dx} + y = e^x, \quad y(1) = 2$$

$$\frac{dy}{dx} + x^{-1}y = x^{-1}e^x$$

$$\int d[xy] = \int e^x dx$$

$$xy = e^x + C$$

$$\text{IF: } e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$$

$$(1)(2) = e^{(1)} + C$$

$$C = 2 - e$$

$$\boxed{y = \frac{e^x + 2 - e}{x} \quad (0, \infty)}$$

viii)  $(x+y)(x+2y) = x^2 + 2xy + y^2$  exact:

$$(x+y)^2 dx + (2xy + x^2 - 1)dy = 0$$

$$My = 2x + 2y$$

$$Nx = 2y + 2x$$

$$fg: 2xy + x^2 - 1$$

↓ wrt y

$$f_x = x^2 + 2xy + y^2$$

↓ wrt x

$$\int (x^2 + 2xy + y^2)dx$$

$$\int (2xy + x^2 - 1)dy$$

$$f(x, y) = xy^2 + x^2y - y + h(x)$$

$$f(x, y) = \frac{1}{3}x^3 + x^2y + y^2x + g(y)$$

$$\boxed{f(x, y) = \frac{1}{3}x^3 + x^2y + y^2x - y = C}$$

Rest on back →

$$ix) \frac{dy}{dx} = (x+y+2)^2 \quad (\text{Substitution})$$

$$u = x + y + 2 \quad \tan^{-1}(u) = x + C$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\boxed{\tan^{-1}(x+y+2) = x+C}$$

$$\frac{du}{dx} = 1 + u^2$$

$$\int \frac{du}{1+u^2} = \int dx$$

5.

$$a) P(t) = P_0 e^{kt}, \quad P(3) = 400, \quad P(10) = 2000$$

$$400 = P_0 e^{k(3)} \rightarrow \ln\left(\frac{400}{P_0}\right) = 3k \rightarrow k = \frac{\ln\left(\frac{400}{P_0}\right)}{3}$$

$$2000 = P_0 e^{\left(\frac{\ln\left(\frac{400}{P_0}\right)}{3}\right)(10)}$$

$$2000 = P_0 \left(\frac{400}{P_0}\right)^{10/3} \rightarrow 2000 = P_0 \cdot \frac{400^{10/3}}{P_0^{10/3}}$$

$$2000 = \frac{400^{10/3}}{P_0^{7/3}} \rightarrow \left[\frac{P_0}{P_0} = \frac{400^{10/3}}{2000}\right]^{3/7}$$

$$P_0 = \frac{400^{10/7}}{2000^{3/7}} = 201$$

$$\boxed{P_0 = 201 \text{ bacteria}}$$

c) 400 gal H<sub>2</sub>O

$$\frac{dA}{dt} = R_{in} - R_{out}$$

$$R_{in} = C_{in} \cdot t_{in}$$

$$R_{out} = C_{out} \cdot t_{out}$$

$$C_{in} = 3 \text{ lb/gal}$$

$$r_{in} = 4 \text{ gal/min}$$

$$r_{out} = 4 \text{ gal/min}$$

$$C_{out} = \frac{A(t)}{400}$$

$$b) \frac{dT}{dt} = k(T-T_m), \quad T(1) = 50, \quad T(0) = 70$$

$$\int \frac{dT}{(T-T_m)} = \int k dt \quad 50 = 20 = 50 e^{k(1)} \\ \ln\left(\frac{30}{50}\right) = k = -0.5108$$

$$\ln|T-T_m| = kt + C_1 \quad @ t = 3 \text{ min:}$$

$$T - T_m = e^{kt} \cdot e^{C_1} \quad T = 50 e^{(-0.5108)(3)} + 20$$

$$T - T_m = C e^{kt}$$

$$70 - 20 = C e^{k(0)} \quad \ln\left(\frac{25-20}{50}\right) = \boxed{t = 4.5 \text{ min}}$$

$$50 = C$$

$$\frac{-0.5108}{-0.5108}$$

$$R_{in} = 3 \text{ lb/gal} \cdot 4 \text{ gal/min} = 12 \text{ lb/min}$$

$$R_{out} = \frac{A(t)}{400} \cdot 4 = \frac{A(t)}{100}$$

$$\frac{dA}{dt} = 12 - \frac{A}{100} \quad \text{IF: } e^{\int \frac{1}{100} dt} = e^{t/100}$$

$$\frac{dA}{dt} + \frac{A}{100} = 12$$

$$\int d[e^{t/100} \cdot A] = \int 12e^{t/100} dt$$

$$e^{t/100} \cdot A = \frac{1}{2} 12e^{t/100} + C$$

$$A = 1200 + Ce^{-t/100} \quad \boxed{A(t) = 1200 - 1100 e^{-t/100}}$$

$$A(0) = 1200 + C e^0 = 100$$

$$\hookrightarrow C = -1100$$

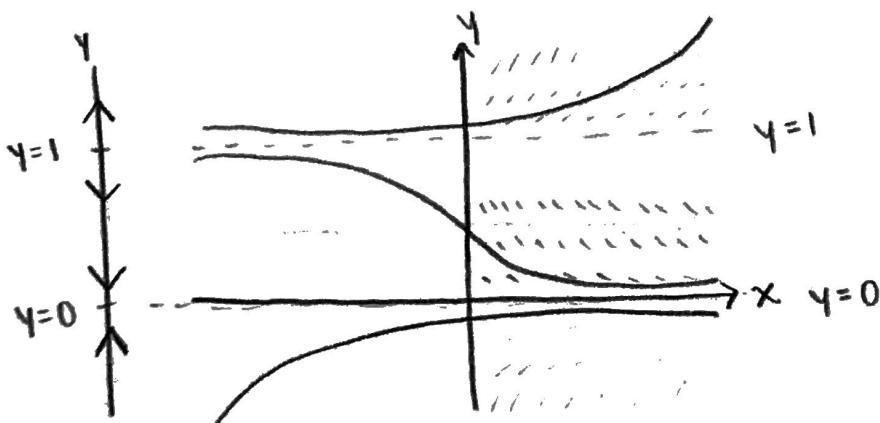
- $y = xe^{2x}$   
 $y' = e^{2x} + x \cdot 2e^{2x} = e^{2x} + 2xe^{2x}$   
 $y'' = 2e^{2x} + (2e^{2x} + 2x \cdot 2e^{2x}) = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x} + 4xe^{2x}$   
 $y'' - 4y' + 4y = 0$   
 $(4e^{2x} + 4xe^{2x}) - 4(e^{2x} + 2xe^{2x}) + 4(xe^{2x}) = 0$   
 ~~$4e^{2x} + 4xe^{2x} - 4e^{2x} - 8xe^{2x} + 4xe^{2x} = 0$~~   
0 = 0 ✓

- $y = C_1 e^{2x} + C_2 x e^{2x}$   
 $y' = 2C_1 e^{2x} + C_2 (e^{2x} + 2xe^{2x}) = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x}$   
 $-1 = C_1 e^{2(0)} + C_2(0)e^{2(0)} \rightarrow \underline{-1 = C_1}$   
 $2 = 2C_1 e^{2(0)} + C_2 e^{2(0)} + 2C_2(0)e^{2(0)} \rightarrow 2 = 2C_1 + C_2$   
 $2 = 2(-1) + C_2$   
 $2 = -2 + C_2$   
 $\underline{4 = C_2}$   
 $y = -1e^{2x} + 4xe^{2x}$

- $\frac{dy}{dx} = y^2 - y$   
 $y^2 - y = 0$   
 $y(y-1) = 0$   
 $\underline{y=0} \quad \underline{y-1=0} \quad \underline{y=1}$ 

Critical points:

$y=0$	asy. stable
$y=1$	unstable



4.

$$(i) \quad x \frac{dy}{dx} = y^2 - y, \quad y(0) = 1$$

Separable

$$\int \frac{1}{y^2-y} dy = \int \frac{1}{x} dx$$

$$\frac{1}{y(y-1)} = \frac{A}{y} + \frac{B}{y-1}$$

$$1 = A(y-1) + By$$

$$y=0: \quad 1 = A(-1)$$

$$-1 = A$$

$$y=1: \quad 1 = B(1)$$

$$1 = B$$

$$1 - \frac{1}{y} = cx$$

$$1 - \frac{1}{(1)} = c(0)$$

$$1 - 1 = 0$$

$$0 = 0$$

$$\int \frac{-1}{y} + \frac{1}{y-1} dy = \int \frac{1}{x} dx$$

$$- \ln|y| + \ln|y-1| = \ln|x| + C_1$$

$$e^{\ln \left| \frac{y-1}{y} \right|} = e^{\ln|x| + C_1}$$

$$\left| \frac{y-1}{y} \right| = \pm x \cdot e^{C_1}$$

$$\frac{y-1}{y} = cx$$

$$(ii) \quad (x^2 + xy + y^2)dx - x^2 dy = 0$$

Homogeneous

$$(x^2 + xux + u^2x^2)dx - x^2(udx + xdu) = 0$$

$$x^2(1+u+u^2)dx - x^2(udx + xdu) = 0$$

$$(1+u+u^2)dx - (udx + xdu) = 0$$

$$(1+u+u^2-u)dx - xdu = 0$$

$$(1+u^2)dx = xdu$$

$$\int \frac{1}{x} dx = \int \frac{1}{1+u^2} du$$

$$\ln|x| = \tan^{-1}u + C$$

$$\begin{aligned} \text{Let } y &= ux & u &= \frac{y}{x} \\ dy &= udx + xdu & \end{aligned}$$

$$\ln|x| = \tan^{-1}\left(\frac{y}{x}\right) + C$$

(iii)  $\frac{dy}{dx} + y^2 \sin x = 0$  Separable

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\int \frac{1}{y^2} dy = \int -\sin x dx$$

$$\frac{-1}{y} = \cos x + C$$

(v)  $\cos x \frac{dy}{dx} + (\sin x)y = 1$  Linear

$$I.F. = e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$$

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

$$\int \sec x \frac{dy}{dx} = \int \sec^2 x dx$$

$$y \sec x = \tan x + C$$

$$y = \frac{\tan x}{\sec x} + \frac{C}{\sec x}$$

$$y = \sin x + C(\cos x)$$

(vi)  $(y^3 - x - y^2 \sin x) dx + (3xy^2 + 2y \cos x) dy = 0$  Exact

$$M_y = N_x \checkmark$$

$$M_y = 3y^2 - 2y \sin x \quad N_x = 3y^2 - \sin x(2y)$$

$$\int (y^3 - x - y^2 \sin x) dx$$

$$= y^3 x - \frac{x^2}{2} + y^2 \cos x + g(y)$$

$$\begin{aligned} & \int (3xy^2 + 2y \cos x) dy \\ &= \frac{3xy^3}{3} + \frac{2y^2 \cos x}{2} + h(x) \\ &= xy^3 + y^2 \cos x + h(x) \end{aligned}$$

$$xy^3 + y^2 \cos x - \frac{x^2}{2} = C$$

$$h(x) = -\frac{x^2}{2}$$

$$g(y) = 0$$

(vii)  $x \frac{dy}{dx} + y = e^x \Rightarrow y(1) = 2$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$$

$$\int x \cdot \frac{dy}{dx} = \int e^x dx$$

$$xy = e^x + C$$

$$y = \frac{e^x}{x} + \frac{C}{x}$$

$$2 = \frac{e^1}{1} + \frac{C}{1}$$

$$2 = e + C$$

$$2 - e = C$$

$$y = \frac{e^x}{x} + \frac{(2-e)}{x}$$

I.F. =  $e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$

Linear

$$y = \frac{e^x + (2-e)}{x}$$

(viii)  $(x+y)^2 dx + (2xy + x^2 - 1) dy = 0$

Exact

$(x^2 + 2xy + y^2) dx + (2xy + x^2 - 1) dy = 0$

$M_y = 2x + 2y \quad N_x = 2y + 2x \quad M_y = N_x$

$\frac{(x+y)(x+y)}{x^2 + 2xy + y^2}$

$$\begin{aligned} & \int (x^2 + 2xy + y^2) dx \\ &= \frac{x^3}{3} + \frac{2yx^2}{2} + y^2x + g(y) \\ &= \frac{x^3}{3} + 4x^2 + y^2x + \underline{g(y)} \end{aligned}$$

$$\begin{aligned} & \int (2xy + x^2 - 1) dy \\ &= \frac{2xy^2}{2} + x^2y - \underline{y} + h(x) \\ & h(x) = \frac{x^3}{3} \\ & g(y) = -y \end{aligned}$$

$$\boxed{\frac{x^3}{3} + 4x^2 + y^2x - y = C}$$

#### 4. summary

- ✓ (i) separable
- ✓ (ii) homogeneous
- ✓ (iii) separable
- (iv) Bernoulli's
- ✓ (v) linear
- ✓ (vi) exact

- ✓ (vii) linear
- ✓ (viii) exact
- (ix) Bernoulli's?  
(Substitution)

$$5(a) \quad P(t) = P_0 e^{kt}$$

$$P(3) = 400 = P_0 e^{3k}$$

$$P(10) = 2000 = P_0 e^{10k}$$

$$\frac{P_0 e^{10k}}{P_0 e^{3k}} = \frac{2000}{400}$$

$$e^{7k} = 5$$

$$7k = \ln 5$$

$$k = \frac{\ln 5}{7}$$

$$3k = \frac{3}{7} \ln 5 = \ln 5^{\frac{3}{7}}$$

$$e^{3k} = 5^{\frac{3}{7}}$$

$$\Rightarrow 400 = P_0 e^{3k}$$

$$400 = P_0 5^{\frac{3}{7}}$$

$$P_0 = \frac{400}{5^{\frac{3}{7}}} = 200.68$$

$$P_0 \approx 201$$

$$\text{Model: } P(t) = 201 e^{1.9932t}$$

5.

$$\text{b) } \frac{dT}{dt} = k(T - T_m)$$

$$T_m = 20^\circ\text{F}$$

$$\int \frac{1}{T - T_m} dT = \int k dt$$

$$e^{\ln|T - T_m|} = e^{kt + C}$$

$$T - T_m = Ce^{kt}$$

$$T(t) = T_m + Ce^{kt}$$

$$T(t) = 20^\circ + Ce^{kt}$$

$$T(0) = 20^\circ + Ce^{k(0)} = 70^\circ$$

$$C = -50^\circ$$

$$T(1) = 20^\circ + 50^\circ e^{k(1)} = 50^\circ$$

$$-50^\circ e^k = -30^\circ$$

$$\ln e^k = \ln(3/5)$$

$$k = -0.5108$$

$$T(t) = 20^\circ + 50^\circ e^{(-0.5108)t} \quad * \text{model}$$

At  $t = 3$  min:

$$T(3) = 20^\circ + 50^\circ e^{(-0.5108)(3)}$$

$$\boxed{T(3) = 30.8^\circ\text{F}}$$

To reach  $25^\circ\text{F}$ :

$$25^\circ = 20^\circ + 50^\circ e^{(-0.5108)t}$$

$$-5^\circ = 50^\circ e^{(-0.5108)t}$$

$$\ln 0.1 = \ln e^{(-0.5108)t}$$

$$\frac{\ln 0.1}{-0.5108} = t$$

$$\boxed{t \approx 4.5 \text{ minutes}}$$

5.

$$\cancel{P(t) = P_0 e^{kt}}$$

$$2000 = 400 e^{k(7)}$$

$$\ln 5 = \ln e^{kt}$$

$$\frac{\ln 5}{7} = k$$

$$0.2291 \approx k$$

$$\cancel{P(3) = 4000} \rightarrow P(0) = 4000$$

$$\cancel{P(10) = 2000} \rightarrow P(7) = 2000$$

c)  $V = 400 \text{ gal}$

$C_{in} = 3 \text{ lb/gal}$

$r_{in} = 4 \text{ gal/min}$

$A(0) = 100 \text{ lb}$

$C_{out} = \frac{A(t)}{400}$

$r_{out} = 4 \text{ gal/min}$

$\frac{dA}{dt} = 12 \text{ lb/min} - 4 \cdot \frac{A(t)}{400}$

$\frac{dA}{dt} + \frac{1}{100} A(t) = 12 \text{ lb/min}$  I.F.  $= e^{\int \frac{1}{100} dt} = e^{\frac{1}{100} \cdot t}$

$\int e^{\frac{1}{100}t} \cdot \frac{dA}{dt} = \int 12 e^{\frac{1}{100}t} dt$

$e^{\frac{1}{100}t} \cdot A = \frac{12 e^{\frac{1}{100}t}}{\frac{1}{100}} + C$

$e^{\frac{1}{100}t} \cdot A = 1200 e^{\frac{1}{100}t} + C$

$A(t) = 1200 + C e^{-\frac{1}{100}t}$

$A(0) = 1200 + C e^{-\frac{1}{100}(0)} = 100 \text{ lb}$

$C = -1100$

$$\boxed{A(t) = 1200 - 1100 e^{-\frac{1}{100}t}}$$

**Exam 2 MTH 335 Spring 2022 Total Pts:100 3/24/2022**

Name: \_\_\_\_\_

Total Received:

Show all work for full credit.

Scan pdf of your solutions and send me @ karna@marshall.edu

1. Find the general solution of the given differential equations. (56 Pts)
  - (a)  $y'' - 3y' - 10y = 0$
  - (b)  $y'' - 3y' + 2y = 0, y(0) = 2, y'(0) = 1$
  - (c)  $y^{(4)} + y''' + y'' = 0$
  - (d)  $y'' + y = 4x \sin(3x)$
  - (e)  $y'' - 4y' + 4y = x^2 - x + 1$
  - (f)  $y'' + 4y = 2 \cos(2x)$
  - (g)  $x^2y'' - 3xy' - 2y = 0$
  - (h)  $x^2y'' + xy' = x^2, y(1) = 1, y'(1) = 0$
2. (8 Pts) Determine a suitable form of  $y_p$  for  $y''' - 4y'' + 4y' = 5x - 6 + 4xe^{2x} - 2e^{5x}$ .  
(Do not solve.)
3. (12 Pts) Use the method of variation of parameter to find the general solution of the differential equation  $y'' + y = \sec x$ .
4. (12 Pts) Solve the given system of differential equation by systematic elimination.
$$\begin{aligned}(D - 2)x + y &= 0 \\ -x + Dy &= 0\end{aligned}$$

5. (12 Pts) The number  $N(t)$  of supermarkets throughout the country that are using computerized check out system is described by the initial-value problem

$$\frac{dN}{dt} = 0.0005N(2000 - N), N(0) = 1.$$

How many companies are expected to adopt new technology when  $t = 10$ ?

## Exam 2

1. a)  $y'' - 3y' - 10y = 0$

AE:  $m^2 - 3m - 10 = 0$   
 $(m-5)(m+2) = 0$   
 $m=5 \quad m=-2$

Gen. Soln.:

$$y = C_1 e^{5x} + C_2 e^{-2x}$$

b)  $y'' - 3y' + 2y = 0 \quad y(0)=2 \quad y'(0)=1$

AE:  $m^2 - 3m + 2 = 0$   
 $(m-2)(m-1) = 0$   
 $m=2 \quad m=1$

$$y = C_1 e^{2x} + C_2 e^x$$

$$y(0)=2 = C_1 e^0 + C_2 e^0$$

$$2 = C_1 + C_2 \rightarrow$$

$$y' = 2C_1 e^{2x} + C_2 e^x$$

$$y'(0)=1 = 2C_1 e^0 + C_2 e^0$$

$$1 = 2C_1 + C_2$$

$$1 = 2(2-C_2) + C_2$$

$$1 = 4 - 2C_2 + C_2$$

$$1 = 4 - C_2$$

$$-3 = -C_2$$

$$3 = C_2$$

$$2 - (3) = C_1$$

$$-1 = C_1$$

Gen. Soln.:

$$y = -1e^{2x} + 3e^x$$

$$2 - C_2 = C_1$$

c)  $y^{(4)} + y''' + y'' = 0$

AE:  $m^{(4)} + m^3 + m^2 = 0$

$$m^2(m^2 + m + 1) = 0$$

$$m=0 \quad m=0 \quad m = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$$

Gen. Soln.:

$$y = C_1 e^{0x} + C_2 x e^{0x} + e^{\frac{1}{2}x} (C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x)$$

$$y = C_1 + C_2 x + e^{\frac{1}{2}x} (C_3 \cos \frac{\sqrt{3}}{2}x + C_4 \sin \frac{\sqrt{3}}{2}x)$$

$$d) \quad y'' + y = 4x \sin(3x)$$

$$AE: \quad m^2 + 1 = 0$$

$$m = \pm i$$

$$y_c = e^{ix} (c_1 \cos x + c_2 \sin x)$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$y_p = (Ax + B) \cos 3x + (Cx + E) \sin 3x$$

$$y_p' = [(Ax + B)(-3 \sin 3x) + (A) \cos 3x] + [(Cx + E)(3 \cos 3x) + (C) \sin 3x]$$

$$y_p'' = [(Ax + B)(-9 \cos 3x) + (A)(-3 \sin 3x) - 3A \sin 3x] + [(Cx + E)(-9 \sin 3x) + (C)(3 \cos 3x) + 3C \cos 3x]$$

$$= [(-9Ax - 9B) \cos 3x + (-3A) \sin 3x + (-3A) \sin 3x] + [(-9Cx - 9E) \sin 3x + (6C) \cos 3x]$$

$$= (-9A)x \cos 3x + (-9B + 6C) \cos 3x + (-6A - 9E) \sin 3x + (-9C)x \sin 3x$$

$$(-9A)x \cos 3x + (-9B + 6C) \cos 3x + (-6A - 9E) \sin 3x + (-9C)x \sin 3x + (Ax + B) \cos 3x + (Cx + E) \sin 3x = 4x \sin 3x$$

$$(-9C + C) = 4$$

$$-8C = 4 \rightarrow C = -\frac{1}{2}$$

$$-9A = 0 \rightarrow A = 0$$

$$(B - 9B + 6C) = 0$$

$$-8B + 6\left(-\frac{1}{2}\right) = 0$$

$$-8B = 3$$

$$B = -\frac{3}{8}$$

$$(-6A - 9E + E) = 0$$

$$-8E = 0$$

$$E = 0$$

Gen. Soln.:

$$y = c_1 \cos x + c_2 \sin x - \frac{3}{8} \cos 3x - \frac{1}{2} x \sin 3x$$

$$e) y'' - 4y' + 4y = x^2 - x + 1$$

$$AE: m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m=2 \quad m=2$$

$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = x^2 - x + 1$$

$$2A - 8Ax - 4B + 4Ax^2 + 4Bx + 4C = x^2 - x + 1$$

$$(4A)x^2 + (-8A+4B)x + (2A-4B+4C) = x^2 - x + 1$$

$$4A = 1 \rightarrow A = \frac{1}{4}$$

$$-8A + 4B = -1$$

$$-8\left(\frac{1}{4}\right) + 4B = -1$$

$$4B = 1 \rightarrow B = \frac{1}{4}$$

$$2A - 4B + 4C = 1$$

$$2\left(\frac{1}{4}\right) - 4\left(\frac{1}{4}\right) + 4C = 1$$

$$-\frac{1}{2} + 4C = 1$$

$$4C = \frac{3}{2} \rightarrow C = \frac{3}{8}$$

Gen. Soln:

$$y = C_1 e^{2x} + C_2 x e^{2x} + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{3}{8}$$

$$f) y'' + 4y = 2\cos(2x)$$

$$AE: m^2 + 4 = 0$$

$$m = \pm 2i$$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

$$y_p = (A \cos 2x + B \sin 2x)x$$

$$y_p' = (A \cos 2x + B \sin 2x) + (-2A \sin 2x + 2B \cos 2x)(x)$$

$$y_p'' = (-2A \sin 2x + 2B \cos 2x) + \left[ (-2A \sin 2x + 2B \cos 2x) + (-4A \cos 2x - 4B \sin 2x)(x) \right]$$

$$= -4A \sin 2x + 4B \cos 2x + (-4A \cos 2x - 4B \sin 2x)(x)$$

$$-4A\sin 2x + 4B\cos 2x + (-4A\cos 2x - 4B\sin 2x)(x) + 4[(A\cos 2x + B\sin 2x)x] = 2\cos(2x)$$

$$(-4A)\sin 2x + (4B)\cos 2x = 2\cos(2x)$$

$$4B = 2 \rightarrow B = \frac{1}{2}$$

$$-4A = 0 \rightarrow A = 0$$

Gen. Soln:

$$y = C_1 \cos 2x + C_2 \sin 2x + (\frac{1}{2} \sin 2x)x$$

$$g) x^2 y'' - 3xy' - 2y = 0$$

$$AE: m(m-1) - 3m - 2 = 0$$

$$m^2 - m - 3m - 2 = 0$$

$$m^2 - 4m - 2 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(1)(-2)}}{2(1)} = \frac{4 \pm \sqrt{24}}{2} = \frac{4 \pm 2\sqrt{6}}{2} = 2 \pm \sqrt{6}$$

Gen. Soln.:

$$y = C_1 x^{(2+\sqrt{6})} + C_2 x^{(2-\sqrt{6})}$$

$$h) x^2 y'' + xy' = x^2 \quad y(1) = 1 \quad y'(1) = 0$$

$$AE: m(m-1) + m = 0$$

$$m^2 - m + m = 0$$

$$m=0 \quad m=0$$

$$y_c = C_1 x^0 + C_2 x^0 \ln x$$

$$y_c = C_1 + C_2 \ln x$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$x^2(2A) + x(2Ax+B) = x^2$$

$$x^2(2A) + x^2(2A) + x(B) = x^2$$

$$4A = 1 \rightarrow A = \frac{1}{4}$$

$$B = 0$$

$$y_p = \frac{1}{4}x^2$$

$$y = C_1 + C_2 \ln x + \frac{1}{4}x^2$$

$$y(1) = 1 = C_1 + C_2 \ln(1) + \frac{1}{4}(1)^2$$

$$\frac{3}{4} = C_1$$

$$y' = C_2 \cdot \frac{1}{x} + \frac{1}{2}x$$

$$y'(1) = 0 = C_2 \cdot \frac{1}{1} + \frac{1}{2}(1)$$

$$-\frac{1}{2} = C_2$$

Gen. Soln.:

$$y = \frac{3}{4} - \frac{1}{2} \ln x + \frac{1}{4}x^2$$

(\* see last page for  $y_p$ )

$$2. \quad y''' - 4y'' + 4y' = 5x - 6 + 4xe^{2x} - 2e^{5x}$$

$$AE: m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m=0 \quad (m-2)(m-2)=0$$

$$m=2 \quad m=2$$

$$y_c = C_1 + C_2 e^{2x} + C_3 x e^{2x}$$

$$y_p = (Ax+B)x + (Cx^3 + Dx)e^{2x} - Fe^{5x}(x)$$

$$3. \quad y'' + y = \sec x$$

AE:  $m^2 + 1 = 0$

$f(x)$

$$m = \pm i$$

$$y_c = C_1 \underbrace{\cos x}_{Y_1} + C_2 \underbrace{\sin x}_{Y_2}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = 0 - \sec x \sin x = -\tan(x)$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = \cos x \cdot \frac{1}{\cos x} = 1$$

$$u_1' = \frac{w_1}{w} = \frac{-\tan(x)}{1} = -\tan(x) \rightarrow u_1 = \ln|\cos(x)|$$

$$u_2' = \frac{w_2}{w} = \frac{1}{1} = 1 \rightarrow u_2 = x$$

$$y_p = (\ln|\cos(x)| \cdot \cos x) + (x \cdot \sin x)$$

$$y_p = \cos x \ln|\cos x| + x \sin x$$

Gen. Soln.:

$$y = C_1 \cos x + C_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

4.

$$\begin{aligned} & \left[ (D-2)x + y = 0 \right] \\ & (D-2) \left[ -x + Dy = 0 \right] \rightarrow \begin{array}{l} (D-2)x + y = 0 \\ + -(D-2)x + (D-2)Dy = 0 \\ \hline (D-2)Dy + y = 0 \\ D^2y - 2Dy + y = 0 \end{array} \\ & AE: m^2 - 2m + 1 = 0 \\ & (m-1)(m-1) = 0 \\ & m=1 \quad m=1 \end{aligned}$$

$$y(t) = c_1 e^t + c_2 t e^t$$

$$x(t) = y'(t) = c_1 e^t + c_2 [t e^t + e^t]$$

$$x(t) = c_1 e^t + c_2 [t e^t + e^t]$$

$$y(t) = c_1 e^t + c_2 t e^t$$

5.

$$\frac{dN}{dt} = 0.0005 N(2000 - N) \quad N(0) = 1$$

$$\begin{array}{ll} 0.0005 N = 0 & 2000 - N = 0 \\ N = 0 & N = 2000 \end{array}$$

$$\int \frac{1}{0.0005 N(2000 - N)} dN = \int dt$$

$$\frac{1}{0.0005 N(2000 - N)} = \frac{A}{0.0005 N} + \frac{B}{2000 - N}$$

$$1 = A(2000 - N) + B(0.0005 N)$$

$$N=0: \quad 1 = A(2000)$$

$$0.0005 = A$$

$$N=2000: \quad 1 = B(0.0005 \cdot 2000)$$

$$1 = B(1)$$

$$1 = B$$

$$\int \frac{0.0005}{0.0005 N} + \frac{1}{2000 - N} dN = \int dt$$

$$\ln|N| - \ln|N-2000| = t + C$$

$$e^{\ln \left| \frac{N}{N-2000} \right|} = e^{t+C}$$

$$\frac{N}{N-2000} = ce^t$$

$$N = ce^t(N-2000)$$

$$N = Nce^t - 2000ce^t$$

$$N - Nce^t = -2000ce^t$$

$$Nce^t - N = 2000ce^t$$

$$N(ce^t - 1) = 2000ce^t$$

$$* N(t) = \frac{2000ce^t}{ce^t - 1}$$

$$N(0) = 1 = \frac{2000ce^{(0)}}{ce^{(0)} - 1}$$

$$1 = \frac{2000c}{c-1}$$

$$c-1 = 2000c$$

$$c - 2000c = 1$$

$$c(1-2000) = 1$$

$$c(-1999) = 1$$

$$c = -0.0005$$

when  $t = 10$ :

$$N(10) = \frac{2000(-0.0005)e^{10}}{(-0.0005)e^{10} - 1}$$

$$N(10) = 1833.52 \approx 1834 \text{ companies}$$

$$1(h) \quad y_c = C_1 + C_2 \ln x$$

$$y'' + \frac{1}{x} y' = 1$$

$$y_1 = 1, \quad y_2 = \ln x, \quad f(x) = 1$$

$$W = \begin{vmatrix} 1 & \ln x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x}$$

$$W_1 = \begin{vmatrix} 0 & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = -\ln x$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$u_1' = \frac{W_1}{W} = \frac{-\ln x}{\frac{1}{x}} = -x \ln x$$

$$\begin{aligned} u_1 &= - \int x \ln x \, dx \\ &= - \left[ \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx \right] \\ &= -\frac{x^2}{2} \ln x + \frac{x^2}{4} \end{aligned}$$

$$u_2' = \frac{W_2}{W} = \frac{1}{\frac{1}{x}} = x$$

$$u_2 = \int x \, dx = \frac{x^2}{2}$$

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{x^2}{2} \ln x + \frac{x^2}{4} + \frac{x^2}{2} \ln x = \frac{x^2}{4}$$

$$\therefore y = y_c + y_p = C_1 + C_2 \ln x + \frac{x^2}{4}$$

$$\begin{aligned} u &= \ln x \rightarrow du = \frac{1}{x} dx \\ dv &= x \, dx \rightarrow v = \frac{x^2}{2} \end{aligned}$$

**Final Exam Part I (A) Math 335 Spring 2022**

Name: Claire Fulks

Received:

*Answer all questions. Show all work for full credit. Bring your solutions tomorrow.*

1. Find two power series solutions of the differential equation  $y'' - xy' - y = 0$  about the ordinary point  $x = 0$ . Form the general solution on  $(-\infty, \infty)$ . (12 Pts)

2. Use the method of Frobenius to obtain two linearly independent series solutions about the regular singular point  $x = 0$  for the differential equation  $3xy'' + (2-x)y' - y = 0$ . (13 Pts)

1.)  $y'' - xy' - y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - x \sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$\downarrow$   
 $n \rightarrow |n+2|$

$$= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1) x^n - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$= c_2(2)(1)x^0 + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1) x^n - \sum_{n=1}^{\infty} c_n n x^n - c_0 x^0 - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$= 2c_2 - c_0 + \sum_{n=1}^{\infty} [c_{n+2}(n+2)(n+1) - c_n n - c_n] x^n = 0$$

$$2c_2 - c_0 = 0$$

$$c_2 = \frac{c_0}{2}$$

$$(n+2)(n+1)c_{n+2} - c_n(n+1) = 0$$

$$c_{n+2} = \frac{c_n(n+1)}{(n+2)(n+1)} = \frac{c_n}{n+2}$$

$$n=1: \quad C_3 = \frac{C_1}{3}$$

$$n=2: \quad C_4 = \frac{C_2}{4} = \frac{\left(\frac{C_0}{2}\right)}{4} = \frac{C_0}{8}$$

$$n=3: \quad C_5 = \frac{C_3}{5} = \frac{\left(\frac{C_1}{3}\right)}{5} = \frac{C_1}{15}$$

$$n=4: \quad C_6 = \frac{C_4}{6} = \frac{\left(\frac{C_0}{8}\right)}{6} = \frac{C_0}{48}$$

$$\begin{aligned} y &= C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 \\ &= C_0 + C_1x + \left(\frac{C_0}{2}\right)x^2 + \left(\frac{C_1}{3}\right)x^3 + \left(\frac{C_0}{8}\right)x^4 + \left(\frac{C_1}{15}\right)x^5 + \left(\frac{C_0}{48}\right)x^6 \end{aligned}$$

$$y = C_0 \left( 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 \right) + C_1 \left( x + \frac{1}{3}x^3 + \frac{1}{15}x^5 \right) \dots$$

$$2.) \quad 3xy'' + (2-x)y' - y = 0$$

$$3xy'' + 2y' - xy' - y = 0$$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

$$3xy'' + 2y' - xy' - y = 3 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1} + 2 \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$= \sum_{n=0}^{\infty} (n+r)(3(n+r-1)+2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1)c_n x^{n+r} = 0$$

$$= \sum_{n=0}^{\infty} (n+r)(3n+3r-1)c_n x^{n+r-1} - \sum_{n=0}^{\infty} (n+r+1)c_n x^{n+r} = 0$$

$$= x^r \left[ \sum_{n=0}^{\infty} (n+r)(3n+3r-1)c_n x^{n-1} - \sum_{n=0}^{\infty} (n+r+1)c_n x^n \right] = 0$$

$n \rightarrow |n+1|$

$$= x^r \left[ \sum_{n=-1}^{\infty} (n+1+r)(3(n+1)+3r-1)c_{n+1} x^n - \sum_{n=0}^{\infty} (n+r+1)c_n x^n \right] = 0$$

$$= x^r \left[ (r)(3r-1)c_0 x^{-1} + \sum_{n=0}^{\infty} (n+1+r)(3n+3+3r-1)c_{n+1} x^n - \sum_{n=0}^{\infty} (n+r+1)c_n x^n \right] = 0$$

$$= x^r \left[ (r)(3r-1)c_0 x^{-1} + \sum_{n=0}^{\infty} [(n+1+r)(3n+3r+2)c_{n+1} - (n+r+1)c_n] x^n \right] = 0$$

$$r(3r-1)c_0 = 0$$

$$r_2 = 0 \quad r_1 = \frac{1}{3}$$

$$(n+1+r)(3n+3r+2)c_{n+1} - (n+r+1)c_n = 0$$

$$c_{n+1} = \frac{(n+r+1)c_n}{(n+1+r)(3n+3r+2)} = \frac{c_n}{3n+3r+2}$$

$$\textcircled{1} \quad r_1 = \frac{1}{3}: \quad C_{n+1} = \frac{C_n}{3n+3}$$

$$\textcircled{2} \quad r_2 = 0: \quad C_{n+1} = \frac{C_n}{3n+2}$$

$$\textcircled{1} \quad n=0: \quad C_1 = \frac{C_0}{3}$$

$$n=1: \quad C_2 = \frac{C_1}{6} = \frac{\left(\frac{C_0}{3}\right)}{6} = \frac{C_0}{18}$$

$$n=2: \quad C_3 = \frac{C_2}{6+3} = \frac{\left(\frac{C_0}{18}\right)}{9} = \frac{C_0}{162}$$

$$\begin{aligned} Y_1 &= x^{1/3} \left( C_0 + C_1 x + C_2 x^2 + C_3 x^3 \right) \\ &= x^{1/3} \left( C_0 + \left(\frac{C_0}{3}\right)x + \left(\frac{C_0}{18}\right)x^2 + \left(\frac{C_0}{162}\right)x^3 \right) \\ &= x^{1/3} \left( 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 \right) \end{aligned}$$

$$\textcircled{2} \quad n=0: \quad C_1 = \frac{C_0}{2}$$

$$n=1: \quad C_2 = \frac{C_1}{3+2} = \frac{\left(\frac{C_0}{2}\right)}{5} = \frac{C_0}{10}$$

$$n=2: \quad C_3 = \frac{C_2}{6+2} = \frac{\left(\frac{C_0}{10}\right)}{8} = \frac{C_0}{80}$$

$$\begin{aligned} Y_2 &= x^0 \left( C_0 + C_1 x + C_2 x^2 + C_3 x^3 \right) \\ &= \left( C_0 + \left(\frac{C_0}{2}\right)x + \left(\frac{C_0}{10}\right)x^2 + \left(\frac{C_0}{80}\right)x^3 \right) \\ &= \left( 1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 \right) \end{aligned}$$

$$Y = C_1 x^{1/3} \left[ 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 \right] + C_2 \left[ 1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{80}x^3 \right]$$

**Final Exam Part II Math 335 Spring 2022 Total Pts:100**

Name: \_\_\_\_\_

Received:

*Answer all questions. Show all work for full credit.*

1. A 16-lb weight stretches a spring 3.2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion if the mass is released from rest at a point 9 in. below equilibrium. (8 pt)
2. A mass of 1 slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at  $t = 0$ , an external force equal to  $f(t) = e^{-t} \sin 4t$  is applied to the system. Find the equation of motion if the surrounding medium offers a damping force numerically equal to 8 times the instantaneous velocity. (10 Pts)
3. Rewrite the given expression using a single power series with general term involving  $x^n$ . (6 Pts)

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$$

4. Use the formula of Laplace transform to find  $\mathcal{L}\{f(t)\}$ . (10 Pts)
  - (a)  $f(t) = 3t^2 + 5t - 6$
  - (b)  $f(t) = 3 \cos 3t - 3 \sin 2t$
  - (c)  $f(t) = 4e^{-3t} + 2t^4$
5. Use appropriate algebra and the formula of inverse Laplace transform to find the following. (10)
  - (a)  $\mathcal{L}^{-1}\left\{\frac{4}{s^2} + \frac{12}{s^5} - \frac{1}{s-3}\right\}$
  - (b)  $\mathcal{L}^{-1}\left\{\frac{-s+12}{s^2+9}\right\}$
  - (c)  $\mathcal{L}^{-1}\left\{\frac{13s-19}{(s-1)(s-3)(s+2)}\right\}$
6. Use the Laplace transform to solve the initial-value problem:  
*(Note: Using other method will not give you any point.)* (8 Pts)

$$y'' - 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

7. Evaluate the following. (15 Pts)
  - (a)  $\mathcal{L}\{e^{-2t} \cos 3t\}$
  - (b)  $\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^4}\right\}$
  - (c)  $\mathcal{L}^{-1}\left\{\frac{s}{s^2-6s+14}\right\}$
  - (d)  $\mathcal{L}\{\sin 2t \mathcal{U}(t-\pi)\}$
  - (e)  $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-1)}\right\}$
8. Use the Laplace transform to solve the initial-value problem  $y' + y = f(t)$ ,  $y(0) = 0$ , (8 Pts)  
 where

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 5, & t \geq 1. \end{cases}$$

**Useful Results:**

- (1)  $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s \rightarrow s-a}$ ,  $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$
- (2)  $\mathcal{L}\{f(t) \mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$ ,
- $\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{L}^{-1}\{F(s)\}|_{t \rightarrow t-a} \mathcal{U}(t-a)$
- (3)  $\mathcal{L}\{y'\} = sY(s) - y(0)$ ,  $\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$

$$F = ks$$

$$(16\text{ lb}) = k(3.2 \text{ ft})$$

$$5 \text{ lb}/\text{ft} = k$$

$$\beta = 1$$

$$w = mg$$

$$(16\text{ lb}) = m(32 \text{ ft/s}^2)$$

$$\frac{1}{2} \text{ slugs} = m$$

$$x(0) = 9 \text{ in} = \frac{3}{4} \text{ ft}$$

$$x'(0) = 0 \text{ ft/s}$$

$$mx'' = -kx - \beta x'$$

$$mx'' + \beta x' + kx = 0$$

$$\frac{1}{2}x'' + x' + 5x = 0$$

$$x'' + 2x' + 10x = 0$$

$$\text{AE: } m^2 + 2m + 10 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$x(0) = \frac{3}{4} = e^0(c_1 \cos 0 + c_2 \sin 0)$$

$$* \frac{3}{4} = c_1$$

$$x'(t) = e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t) - e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$x'(0) = 0 = e^0(-3c_1 \sin 0 + 3c_2 \cos 0) - e^0(c_1 \cos 0 + c_2 \sin 0)$$

$$0 = 3c_2 - c_1$$

$$\frac{3}{4} = 3c_2$$

$$* \frac{1}{4} = c_2$$

$$x(t) = e^{-t}\left(\frac{3}{4} \cos 3t + \frac{1}{4} \sin 3t\right)$$

$$2. \quad m = 1 \text{ slug}$$

$$s = 2 \text{ ft}$$

$$\beta = 8$$

$$mg = ks$$

$$(1)(32 \text{ ft/s}^2) = k(2 \text{ ft})$$

$$16 \frac{\text{lb}}{\text{ft}} = k$$

$$mx'' + \beta x' + kx = f(t)$$

$$x'' + 8x' + 16x = e^{-t} \sin 4t$$

$$\text{AE: } m^2 + 8m + 16 = 0$$

$$(m+4)(m+4) = 0$$

$$m = -4 \quad m = -4$$

$$x(0) = 0 \text{ ft}$$

$$x'(0) = 0 \text{ ft/s}$$

$$x(t) = C_1 e^{-4t} + C_2 t e^{-4t}$$

$$x'' + 8x' + 16x$$

$$x_p = A e^{-t} \cos 4t + B e^{-t} \sin 4t$$

$$x_p' = A [e^{-t}(-4\sin 4t) - e^{-t}(\cos 4t)] + B [e^{-t}(4\cos 4t) - e^{-t}(\sin 4t)]$$

$$x_p'' = A \{e^{-t}(-16\cos 4t) - e^{-t}(-4\sin 4t)\} - \{e^{-t}(-4\sin 4t) - e^{-t}(\cos 4t)\} \\ + B \{e^{-t}(-16\sin 4t) - e^{-t}(4\cos 4t)\} - \{e^{-t}(4\cos 4t) - e^{-t}(\sin 4t)\}$$

$$= A [e^{-t}(-15\cos 4t) - e^{-t}(-8\sin 4t)] + B [e^{-t}(-15\sin 4t) - e^{-t}(8\cos 4t)]$$

$$A [e^{-t}(-15\cos 4t) - e^{-t}(-8\sin 4t)] + B [e^{-t}(-15\sin 4t) - e^{-t}(8\cos 4t)]$$

$$+ 8 \left[ A [e^{-t}(-4\sin 4t) - e^{-t}(\cos 4t)] + B [e^{-t}(4\cos 4t) - e^{-t}(\sin 4t)] \right]$$

$$+ 16 \left[ A e^{-t} \cos 4t + B e^{-t} \sin 4t \right] = e^{-t} \sin 4t$$

$$(-15A)e^{-t} \cos 4t + (8A)e^{-t} \sin 4t (-15B)e^{-t} \sin 4t (-8B)e^{-t} \cos 4t$$

$$(-32A)e^{-t} \sin 4t (-8A)e^{-t} \cos 4t + (32B)e^{-t} \cos 4t (-8B)e^{-t} \sin 4t$$

$$+ (16A)e^{-t} \cos 4t + (16B)e^{-t} \sin 4t = e^{-t} \sin 4t$$

$$e^{-t} \sin 4t (8A - 15B - 32A - 8B + 16B) + e^{-t} \cos 4t (-15A - 8B - 8A + 32B + 16A)$$

$$e^{-t} \sin 4t (-24A - 7B) + e^{-t} \cos 4t (-7A + 24B) = e^{-t} \sin 4t = e^{-t} \sin 4t$$

$$-24A - 7B = 1$$

$$-7A + 24B = 0$$

$$24B = 7A$$

$$B = \frac{7}{24}A$$

$$-24A - 7\left(\frac{7}{24}A\right) = 1$$

$$-24A - \frac{49}{24}A = 1$$

$$-\frac{625}{24}A = 1$$

$$* A = \frac{-24}{625}$$

$$B = \frac{7}{24} \left( \frac{-24}{625} \right)$$

$$* B = -\frac{7}{625}$$

$$x_p = \frac{-24}{625} e^{-t} \cos 4t - \frac{7}{625} e^{-t} \sin 4t$$

$$x(t) = C_1 e^{-4t} + C_2 t e^{-4t} - \frac{24}{625} e^{-t} \cos 4t - \frac{7}{625} e^{-t} \sin 4t$$

$$x(0) = 0 = C_1 e^0 + C_2(0)e^0 - \frac{24}{625} e^0 \cos 0 - \frac{7}{625} e^0 \sin 0$$

$$0 = C_1 - \frac{24}{625}$$

$$+ \frac{24}{625} = C_1$$

$$x'(t) = -4C_1 e^{-4t} + C_2 \left[ t(-4e^{-4t}) + e^{-4t} \right] - \frac{24}{625} \left[ e^{-t}(-4\sin 4t) - e^{-t}(\cos 4t) \right] \\ - \frac{7}{625} \left[ e^{-t}(4\cos 4t) - e^{-t}(\sin 4t) \right]$$

$$x'(0) = 0 = -4C_1 e^0 + C_2 \left[ 0 + e^0 \right] - \frac{24}{625} \left[ 0 - e^0 \cos 0 \right] - \frac{7}{625} \left[ e^0 4\cos 0 \right]$$

$$0 = -4C_1 + C_2 + \frac{24}{625} - \frac{28}{625}$$

$$0 = -4 \left( \frac{24}{625} \right) + C_2 - \frac{4}{625}$$

$$\frac{4}{625} = \frac{-96}{625} + C_2$$

$$\frac{100}{625} = C_2$$

$$+ \frac{4}{25} = C_2$$

$$x(t) = \frac{24}{625} e^{-4t} + \frac{4}{25} t e^{-4t} - \frac{24}{625} e^{-t} \cos 4t - \frac{7}{625} e^{-t} \sin 4t$$

3.

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$$

$n \rightarrow \downarrow n+2 \qquad n \rightarrow \downarrow n-2$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} c_{n-2} x^n$$

$$(2)(1)c_2 x^0 + (3)(2)c_3 x^1 + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + c_{n-2}] x^n$$

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} + c_{n-2}] x^n$$

4. a)  $L\{3t^2 + 5t - 6\} = 3L\{t^2\} + 5L\{t\} - 6L\{1\}$

$$= 3\left[\frac{2!}{s^3}\right] + 5\left[\frac{1!}{s^2}\right] - 6\left[\frac{1}{s}\right]$$

$= \frac{6}{s^3} + \frac{5}{s^2} - \frac{6}{s}$

b)  $L\{3\cos 3t - 3\sin 2t\} = 3L\{\cos 3t\} - 3L\{\sin 2t\}$

$$= 3\left[\frac{s}{s^2 + 3^2}\right] - 3\left[\frac{2}{s^2 + 2^2}\right]$$

$= \frac{3s}{s^2 + 9} - \frac{6}{s^2 + 4}$

c)  $L\{4e^{-3t} + 2t^4\} = 4L\{e^{-3t}\} + 2L\{t^4\}$

$$= 4\left[\frac{1}{s+3}\right] + 2\left[\frac{4!}{s^5}\right]$$

$= \frac{4}{s+3} + \frac{48}{s^5}$

5. a)  $L^{-1}\left\{\frac{4}{s^2} + \frac{12}{s^5} - \frac{1}{s-3}\right\}$

$$= 4L^{-1}\left\{\frac{1}{s^2}\right\} + 12L^{-1}\left\{\frac{1}{s^5}\right\} - L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \frac{4}{1!}L\left[\frac{1!}{s^2}\right]_{n=1}^{n=1} + \frac{12}{4!}L\left[\frac{4!}{s^5}\right]_{n=4}^{n=4} - e^{3t}$$

$$= 4[t] + \frac{1}{2}[t^4] - e^{3t}$$

$= 4t + \frac{1}{2}t^4 - e^{3t}$

b)  $L^{-1}\left\{\frac{-s+12}{s^2+9}\right\} = L^{-1}\left\{\frac{-s}{s^2+9}\right\} + L^{-1}\left\{\frac{12}{s^2+9}\right\}$

 $= -1 L^{-1}\left\{\frac{s}{s^2+9}\right\} + 4L^{-1}\left\{\frac{3}{s^2+9}\right\}$ 
 $= -1 \left[ \cos 3t \right] + 4 \left[ \sin 3t \right]$ 

$= -\cos 3t + 4\sin 3t$

c)  $L^{-1}\left\{\frac{13s-19}{(s-1)(s-3)(s+2)}\right\} = L^{-1}\left\{\frac{-3}{s+2} + \frac{1}{s-1} + \frac{2}{s-3}\right\}$  PFD solver ↓

 $= -3 L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} + 2 L^{-1}\left\{\frac{1}{s-3}\right\}$ 
 $= -3 \left[ e^{-2t} \right] + e^t + 2 \left[ e^{3t} \right]$ 

$= -3e^{-2t} + e^t + 2e^{3t}$

6.  $y'' - 5y' + 4y = 0 \quad y(0) = 1 \quad y'(0) = 0$

$L\{y'' - 5y' + 4y\} = L\{0\}$

$s^2Y(s) - sy(0) - y'(0) - 5[sY(s) - y(0)] + 4[Y(s)] = 0$

$s^2Y(s) - s - 5sY(s) + 5 + 4Y(s) = 0$

$Y(s)[s^2 - 5s + 4] = s - 5$

$Y(s) = \frac{s-5}{s^2 - 5s + 4} = \frac{s-5}{(s-4)(s-1)}$

$Y(s) = \frac{4/3}{s-1} - \frac{1/3}{s-4}$  PFD solver

$y(t) = L^{-1}\left\{\frac{4/3}{s-1} - \frac{1/3}{s-4}\right\}$

$= \frac{4}{3}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s-4}\right\}$

$= \frac{4}{3}[e^t] - \frac{1}{3}[e^{4t}] = \frac{1}{3}e^t - \frac{1}{3}e^{4t}$

$$\begin{aligned}
 7. \text{ a) } L\{e^{-2t} \cos 3t\} &= L\{\cos 3t\} \Big|_{s \rightarrow s+2} \\
 &= \frac{s}{s^2 + 3^2} \Big|_{s \rightarrow s+2} = \frac{(s+2)}{(s+2)^2 + 9} \\
 &= \frac{s+2}{(s^2 + 4s + 4) + 9} = \boxed{\frac{s+2}{s^2 + 4s + 13}}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } L^{-1}\left\{\frac{6}{(s-2)^4}\right\} &= 6L^{-1}\left\{\frac{1}{(s-2)^4}\right\} = 6L^{-1}\left\{\frac{1}{s^4}\right\} \Big|_{s \rightarrow s-2} \\
 &= \frac{6}{3!} L^{-1}\left\{\frac{3!}{s^4}\right\} \Big|_{s \rightarrow s-2} = \frac{6}{6} t^3 e^{2t} = \boxed{t^3 e^{2t}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c)} & L^{-1}\left\{\frac{s}{s^2 - 6s + 14}\right\} \\
 &= L^{-1}\left\{\frac{s}{(s^2 - 6s + 9) + 5}\right\} \\
 &= L^{-1}\left\{\frac{s}{(s-3)^2 + 5}\right\} \\
 &= L^{-1}\left\{\frac{s-3+3}{s^2+5}\right\} \Big|_{s \rightarrow s-3} \quad k=\sqrt{5} \\
 &= \boxed{\cos \sqrt{5}t}
 \end{aligned}$$

$$\begin{aligned}
 \text{d) } L\{\sin 2t u(t-\pi)\} &= e^{-\pi s} L^{-1}\{\sin(2t+\pi)\} \\
 &= e^{-\pi s} \left[ L^{-1}\{\sin 2t\} + L^{-1}\{\sin \pi\} \right] \\
 &= e^{-\pi s} \left[ \frac{2}{s^2 + 4} + \sin \pi L^{-1}\{1\} \right] \\
 &= \boxed{e^{-\pi s} \left[ \frac{2}{s^2 + 4} + \frac{\sin \pi}{s} \right]}
 \end{aligned}$$

$$e)^* L^{-1} \left\{ \frac{e^{-s}}{s(s-1)} \right\} = L \{ u(t-a) \} = \frac{e^{-as}}{s}$$

= u(t-1)

\* 8.  $y' + y = f(t)$   $y(0) = 0$

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 5, & t \geq 1 \end{cases}$$

$$L \{ y' + y \} = L \{ 5u(t-1) \}$$

$$sY(s) - y(0)^0 + Y(s) = e^{-s} \frac{5}{s}$$

$$Y(s)[s+1] = \frac{5e^{-s}}{s}$$

$$Y(s) = e^{-s} \frac{5}{s(s+1)}$$

$$Y(s) = e^{-s} \left[ \frac{5}{s} - \frac{5}{s+1} \right]$$

$$y(t) = L^{-1} \left\{ \left[ \frac{5}{s} - \frac{5}{s+1} \right] e^{-s} \right\}$$

$$= u(t-1) L^{-1} \left\{ \frac{5}{s} - \frac{5}{s+1} \right\} \Big|_{t \rightarrow t-1}$$

$$= u(t-1) [5 - 5e^{-t}] \Big|_{t \rightarrow t-1}$$

$$y(t) = u(t-1) (5 - 5e^{-t+1})$$